FINANCIAL ENGINEERING
KBP FFŪ

# Financial Derivatives II
# Part 1

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EVROPSKÁ UNIE
Evropské strukturální a investiční fondy
Operační program Výzkum, vývoj a vzdělávání

MINISTERSTVO ŠKOLSTVÍ, MLÁDEŽE A TĚLOVÝCHOVY

# Course Organization

- Midterm Test: 6-7 week, max. 25 pts
- 2-3 projects, valuation of derivative contracts, max. 30 pts., presentations of selected solutions
- Final Test – TBA (last lecture or during the exam period), max. 45 pts.
  - Excellent: 90-100
  - Very good: 75-89
  - Good: 60-74
- The Final can be repeated only if the total score is at least 50
- Slides, project assignment, and scores on ISIS

# Literature

| Requirement | ISBN | Title | Author | Year of Publication |
|---|---|---|---|---|
| Required | 978-80-245-1980-7 | Financial Derivatives – Valuation , Hedging and Risk Management | Witzany, Jiří | 2013, Oeconomica |
| Alternatively | 978-80-245-1878-7 | Financial Derivatives and Market Risk Management, Part II | Witzany, Jiří | 2012, Oeconomica |
| Recommended[1] | 978-0-13-216494-8 | Options, Futures, and Other Derivatives, 841 p. | Hull, John C. | 2012, 8th edition, Pearson |
| Recommended | 978-80-245-1811-4 | Financial Derivatives and Market Risk Management, Part I | Witzany, Jiří | 2011, Oeconomica |
| Optional | 0387249680, 0387401016 | Stochastic Calculus for Finance I,II | Steven, E. Shreve | 2004-5 |
| Optional | 978-80-7431-079-9 | Matematika cenných papírů. 288 s. | Cipra, Tomáš | 2013 |
| Optional | 978-0-470-01870-5 | Paul Wilmott on Quantitative Finance | Paul Wilmott | Wiley, 2006 |

*Source: Author*

1) The course should cover chapters 5-8 from Witzany (2013) or Chapters 19-31 from John Hull, 8th Edition (2012)

# Content

- Introduction – overview of B.-S. option pricing and hedging
- Market Risk Management and Measurement
- Estimating volatilities and correlations
- Interest Rate Derivatives Pricing-Martingale and measures
- Standard Market Model

# Content

- Convexity, time, and quanto adjustments
- Short-rate and advanced interest rate models
- Volatility smiles
- Exotic options
- Alternative stochastic models
- Numerical methods for option pricing
- Credit derivatives

# Introduction – an overview from Financial Derivatives I Forwards, Futures, Swaps, and Options

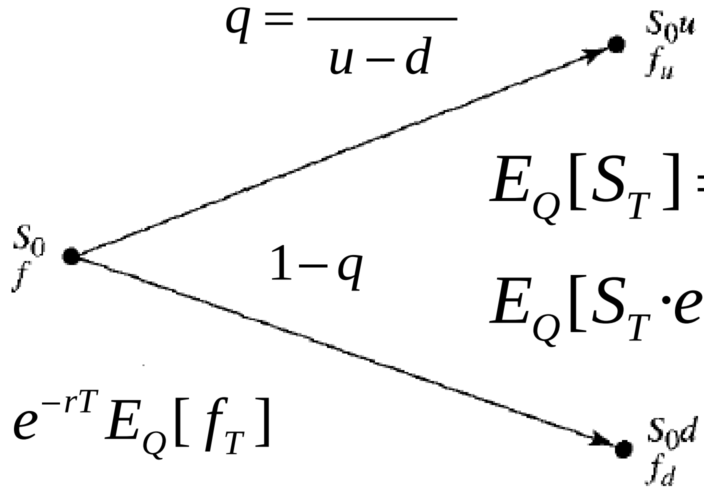
- Forwards, futures, and swaps are unconditional derivatives while options are conditional
- Call/Put, long/short position, European/American, in/at/out of the money, intrinsic/time value
- Stock options – mostly exchange traded – CBOE, PHLX, AMEX, PACIFEX, EUREX
- Options on indices – exchange traded – cash settlement
- Currency options - exchange traded and OTC
- Options on futures contracts – exchange traded – options to acquire long or short position in a futures contract
- Expiration date and strike price is defined – options are traded at a premium

# General one-step binomial tree

- A riskless portfolio can be set-up in general
- Risk-neutral probabilities can be calculated as follows

$$q = \frac{e^{rT} - d}{u - d}$$

$$\Delta = \frac{f_u - f_d}{S_0 u - S_0 d} \approx \frac{\partial f}{\partial S}$$




$$E_Q[S_T] = S_0 \cdot e^{rT}$$

$$E_Q[S_T \cdot e^{-rT} \mid 0] = S_0 \cdot e^{-r0}$$

$$f = e^{-rT}(q \cdot f_u + (1-q) \cdot f_d) = e^{-rT} E_Q[f_T]$$

*Source: Author*

# No-Arbitrage approach

- Consider a stock with current value $S_0$, which can move over period $T$, either to $S_{up}$ or $S_{down}$
- We want to **calculate the value of call option $f$** with strike price $K$ and maturity $T$
- We can construct a riskless portfolio consisting of **1 short call option** and Δ **long stocks**
- The value of the portfolio now is: $P_0 = \Delta S_0 - f_0$
- In order for the portfolio to be riskelss, it must hold that:
- $\Delta S_{Up} - \max\left(S_{Up} - K, 0\right) = \Delta S_{Down} - \max(S_{Down} - K, 0)$, or:
- $\Delta S_{Up} - f_{Up} = \Delta S_{Down} - f_{Down} = \Delta S_T - f_T = P_T$
- We can thus calculate the value of Δ as: $\Delta = \frac{f_{Up} - f_{Down}}{S_{Up} - S_{Down}}$
- The riskless portfolio must earn the riskfree interest rate: $P_T = P_0 e^{rT}$
- Thus: $P_T = e^{rT}(\Delta S_0 - f_0)$
- The option price $f_0$ is the only unknown, so we can easily calculate it

# Risk-neutral Valuation

- Note, that the result does not depend on probabilities of the two scenarios
- However if we set up *q*, probability of the movement up, so that $E(S_T)=S_0e^{RT}$ (the stock return equals to the risk-free rate) then it turns out that the price of the option equals to the discounted expected pay-off
- *Risk neutral valuation principle: we can assume that the world is risk neutral when pricing an option. The result is valid also in the real world which is not risk-neutral!*

# Risk-Neutral approach

- In the risk-neutral world:
  - Asset prices grow on average with the risk-free interest rate
  - We can compute the value of any derivative at time *t*, by calculating the risk-neutral expected value at time *T* and then discounting it with the risk-free rate to time *t*
- We will further denote risk-neutral expectations as $E_Q$
- In the risk-neutral world it holds that: $E_q(S_T) = S_0 e^{rT}$
- In the binomial tree setting this means that:
- $S_0 = e^{-rT}[q * u * S_0 + (1 - q) * d * S_0]$
- Where $u$ and $d$ denote the potential increase/decrease and $q$ is the risk-neutral probability of increase
- The risk-neutral probability is then: $q = \frac{e^{rT} - d}{u - d}$
- And option can be valued as: $f_0 = e^{-rT}[q f_u + (1 - q) f_d]$

# Two-step binomial trees

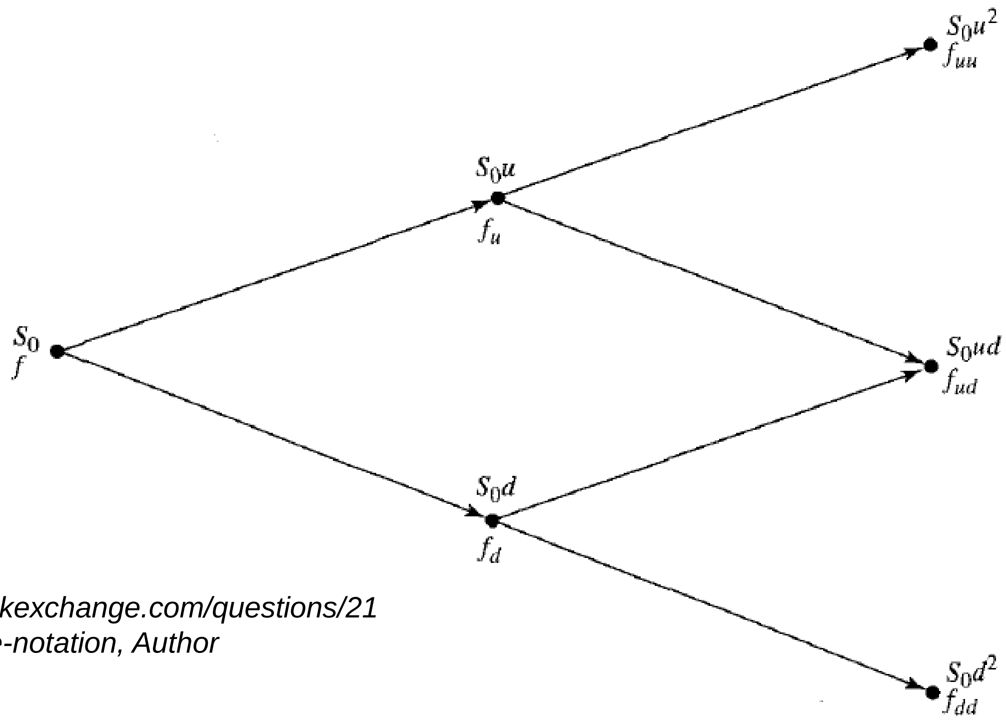



**Figure 10.6** Stock and option prices in a general two-step tree

The binomial tree is recombining

*Source: https://quant.stackexchange.com/questions/21773/binomial-tree-notation, Author*

The option price equals to the discounted expected pay-off in a risk neutral world $f = e^{-rT}\hat{E}[f_T]$

# General risk-neutral probabilities

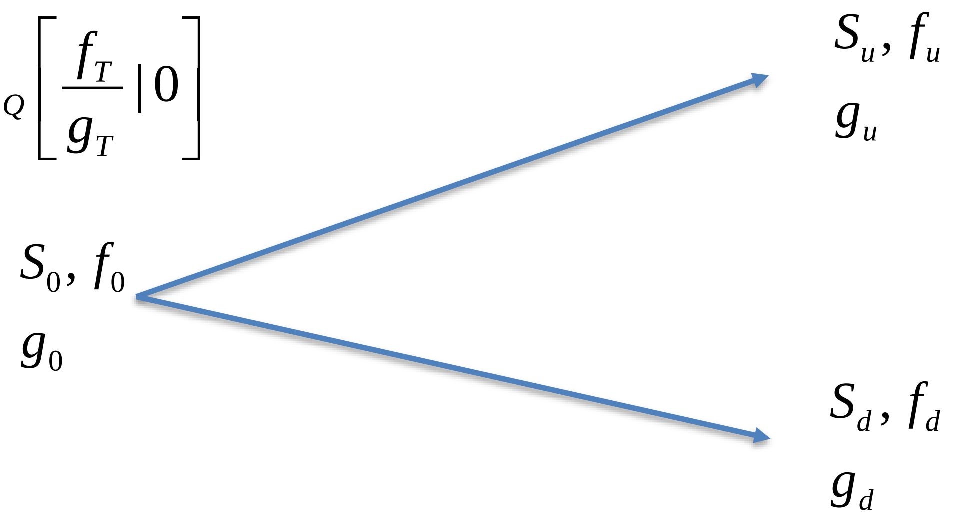
- A general discount factor $g$ … **numeraire**
- Define the risk neutral probability so that

$$Z_0 = qZ_u + (1-q)Z_d \text{ where } Z = \frac{S}{g} \qquad g = e^{-rt}$$

- And use the replication argument to show that *f/g* is a **martingale**

$$\frac{f_0}{g_0} = q\frac{f_u}{g_u} + (1-q)\frac{f_d}{g_d} = \mathrm{E}_Q\left[\frac{f_T}{g_T} \mid 0\right]$$

$$\text{as } f = \alpha S + \beta g$$




*Source: Author*

# Continuous Stochastic Modeling of Stock Prices

- **Stochastic proces** – variable $x_t$ whose value depends on time $t$ and changes in an uncertain way
- $f(x_t)$ is the distribution of variable $x_t$ at time $t$
- Discrete/continuous time, discrete/continuous variable
- **Markov proces** – only the present value of a variable is relevant for the future: $f(x_{t+h}|x_t, \dots, x_0) = f(x_{t+h}|x_t)$
- Market rates (stock prices, exchange rates, interest rates, etc.) are usually assumed to follow the Markov proces (vs. Technical analysis)
- **Martingale** – the expected value of the variable in the future is equal to the current value $\mathrm{E}(x_{t+h}|x_t, \dots, x_0) = x_t$

# Wiener process

- **Continuous time proces** – any time period can be divided into arbitrary number of steps
- **Wiener proces (Brownian motion)** – Continuous Markov proces, where $z(1) - z(0)$ has distribution $N(0,1)$ and the distributions are uniform and independent for small time steps
- **Properties of the Wiener proces:**
  1. $z(0) = 0$
  2. $z$ has independent increments
  3. $z(t + u) - z(t) \sim N(0, u)$ (i.e. normal distribution with variance $u$)
  4. $z$ has continous paths with probability 1
- **Stochastic difference equation:** $dz = \varepsilon\sqrt{dt}$ where $\varepsilon$ is random variable with the distribution $N(0,1)$
- Square root of time rule: $z(T_0 + t) - z(T_0)$ has distribution $N(0, t)$
- **Generalized Wiener proces:** $dx = adt + bdz$
- i.e. $x(T_0 + t) - x(T_0)$ has the distribution $N(at, b^2 t)$

# Wiener proces illustration

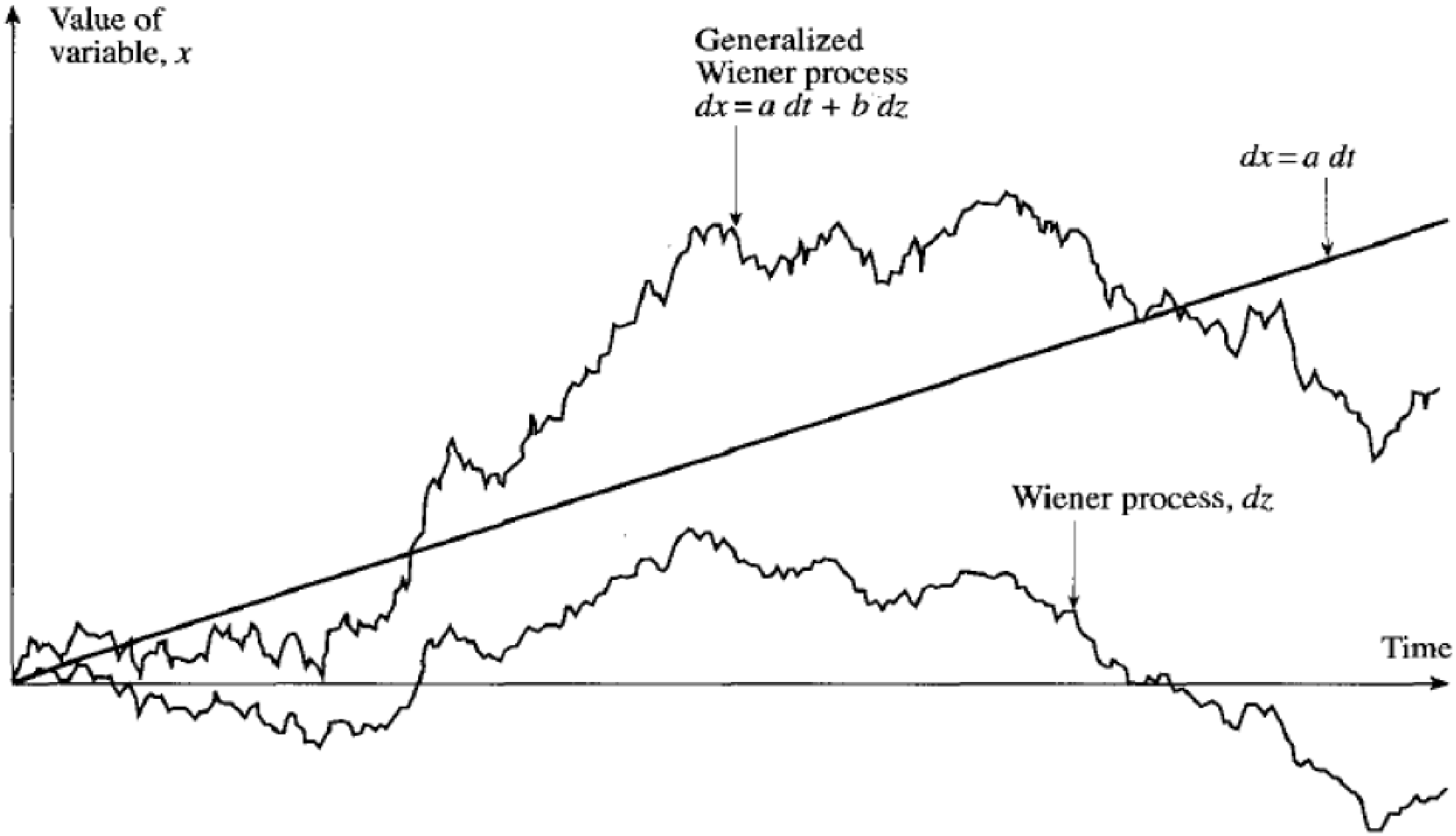



**Figure 11.2** Generalized Wiener process: $a = 0.3$, $b = 1.5$

Source: John Hull, Options, Futures, and Other Derivatives, 5th edition

# Ito‘s process

- $dx = a(x,t)dt + b(x,t)dz$, where $dz$ is the *Wiener process* increment
- The drift and the variance depend on x and t
- The process for Stock prices *S*: normally distributed annualized rate of return with expected value $\mu$ and standard deviation $\sigma$ (observed for a small time periods, not one year)
- **Geometric Brownian Motion**

$$dS = \mu S dt + \sigma S dz$$

- It turns out that $S(T)$ is not normally distributed, but lognormally distributed

# Ito‘s Lemma

- If $G = G(x,t)$ where x follows an Ito‘s process

$$dx = a(x,t)dt + b(x,t)dz,$$

then $G$ follows the Ito‘s process

$$dG = \left( \frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2 \right) dt + \frac{\partial G}{\partial x} b dz$$

where $dz$ is the same as above.

# Derivation of Ito‘s Lemma

- Ito‘s Lemma can be derived from Taylor series expansion
- Let $x$ follow an Ito‘s proces:
- $dx = a(x,t)dt + b(x,t)dz$
- And let $G$ be a function $G = G(x,t)$
- The Taylor series expansion of $G$ would be:
- $dG = \frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial t}dt + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}dx^2 + \cdots$
- Substituting for $dx$ we get:
- $dG = \frac{\partial G}{\partial x}(adt + bdz) + \frac{\partial G}{\partial t}dt + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}(a^2dt^2 + 2abdtdz + b^2dz^2)$
- It holds that $dt^2 \approx 0$ and $dtdz \approx 0$, but $dz^2 = dt$
- By rewriting the formula we get the Ito‘s Lemma:
- $dG = \left(\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2\right)dt + \frac{\partial G}{\partial x}bdz$

# Application of Ito‘s Lemma

- Let $S$ follow a Geometric Brownian Motion:
- $dS = \mu S dt + \sigma S dz$
- And let $G$ be a function $G = \ln(S)$
- Process followed by $G$ can be derived with Ito‘s Lemma:
- $dG = \left(\frac{\partial G}{\partial S}\mu S + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\sigma^2 S^2\right)dt + \frac{\partial G}{\partial S}\sigma S dz$
- We can compute the derivatives as:
- $\frac{\partial G}{\partial S} = \frac{1}{S} \qquad \frac{\partial G}{\partial t} = 0 \qquad \frac{\partial^2 G}{\partial S^2} = -\frac{1}{S^2}$
- By substituting for the derivatives, we can get:
- $dG = \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dz$

# Lognormal property

- Let $G = \ln S$, where $S$ follows the geometric Brownian motion, then applying the Ito's lemma we get:

$$dG = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dz$$

- Which is a Generalized Wiener Process
- Consequently $\ln S(T)$-$\ln S(0)$ has the normal distribution $N((\mu\text{-}\sigma^2/2)T, \sigma^2 T)$

# Lognormal property

- Lognormal property of stock prices

$$dS = \mu S dt + \sigma S dz$$

$$\frac{dS}{S} \approx N\left(\mu dt, \sigma^2 dt\right)$$

$$\ln S_T - \ln S_0 \approx N\left((\mu - \sigma^2/2)T, \sigma^2 T\right)$$

$$\ln S_T \approx N\left(\ln S_0 + (\mu - \sigma^2/2)T, \sigma^2 T\right)$$

# Lognormal property of Stock prices

- It can be shown that

$$E(S_T) = S_0 \; e^{\mu T}$$

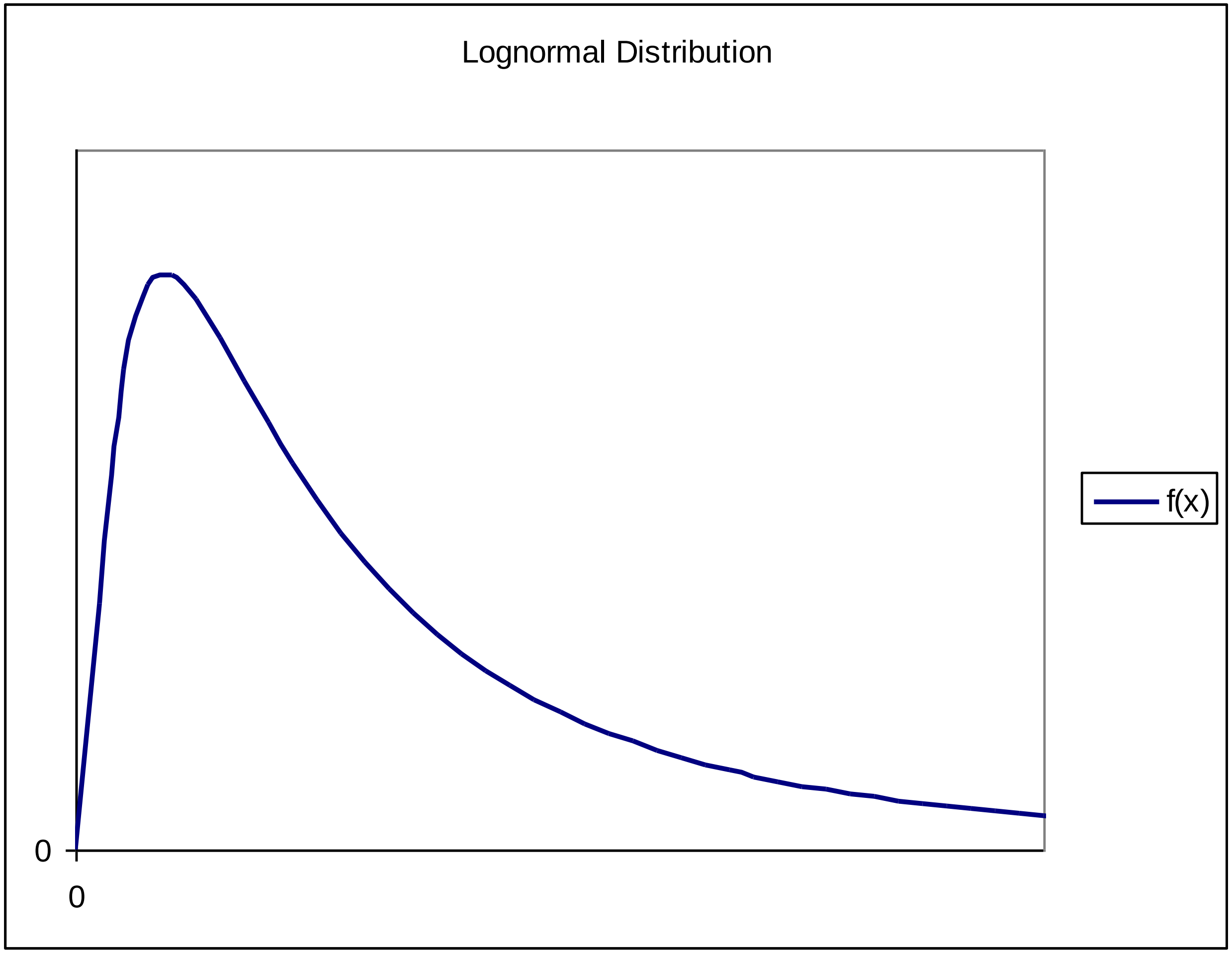
$$\text{var}(S_T) = S_0^{\;2} e^{2\mu T}(e^{\sigma^2 T} - 1)$$

- Stock price can be modelled/simulated as

$$S_T = S_0 e^{\eta T}$$

$$\eta \approx N\left(\mu - \frac{\sigma^2}{2}, \frac{\sigma^2}{T}\right)$$

# Lognormal distribution




*Source: https://www.vosesoftware.com/riskwiki/LognormalBdistribution.php, Author*

# Assumptions of the Black-Scholes Model

1. The asset price follows the geometric Brownian motion process with constant drift and volatility (lognormal returns): $dS = \mu S dt + \sigma S dz$
2. There is no income paid by the asset.
3. The risk free interest rate *r* is constant. We can lend and borrow at the same rate and without any restrictions.
4. There are no transaction costs and taxes.
5. Assets are arbitrarily divisible.
6. Short selling of securities is possible without restrictions.
7. There are no arbitrage opportunities.
8. Security trading is continuous; i.e., we can trade in infinitesimal time interval.

# Black-Scholes differential equation

$$dS = \mu S dt + \sigma S dz$$

Ito‘s lemma applied to $f = f(S,t)$ :

$$df = \left( \frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 \right) dt + \frac{\partial f}{\partial S}\sigma S dz$$

Risk - less portfolio: - 1 option + Δ stocks, i.e.

$$\Pi = -f + \frac{\partial f}{\partial S} S$$

# Black-Scholes differential equation

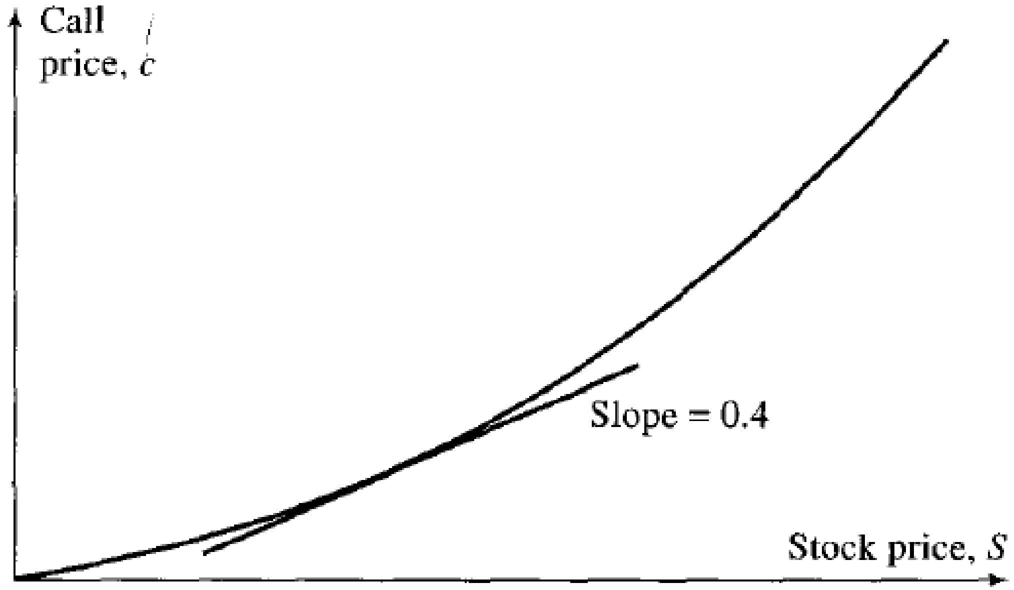



**Figure 12.2** Relationship between $c$ and $S$

Source: John Hull, Options, Futures, and Other Derivatives, 5th edition

# Black-Scholes differential equation

Combing the SDEs for *dS* and *df*

$$d\Pi = -df + \frac{\partial f}{\partial S} dS =$$

$$= -\left( \frac{\partial f}{\partial S} \mu S + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) dt - \frac{\partial f}{\partial S} \sigma S dz + \frac{\partial f}{\partial S} \mu S dt + \frac{\partial f}{\partial S} \sigma S dz =$$

$$= -\left( \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) dt$$

# Black-Scholes differential equation

Hence the portfolio is risk less, as expected, and so

$$d\Pi = r\Pi dt$$

$$-\left(\frac{\partial f}{\partial t}+\frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)dt = r\Pi dt = r(-f+\frac{\partial f}{\partial S}S)dt$$

And we get the Black-Scholes-Merton partial differential equation:

$$\frac{\partial f}{\partial t}+\frac{\partial f}{\partial S}rS+\frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 = rf$$

# Risk-neutral valuation

- The Black-Scholes-Merton equation does not depend on the expected return , i.e. on investors risk preferences!!!
- We can assume that we are in a risk-neutral world.
- The result will be the same as in the real world with risk sensitive investors.
- Consequently we can simply discount the expected pay-off of an option using the risk free rate.
- The result will be the unique solution of the differential equation

# The Black-Scholes Formula

- Consider a European Call option
- Then in the risk neutral world, where $dS = rSdt + \sigma Sdz$ the value of the option at is:

$$c_0 = e^{-rT}\hat{E}[c_T] = e^{-rT}\hat{E}[\max(S_T - K, 0)]$$

- Using the lognormal property of $S_T$ we obtain

$$c_0 = S_0 N(d_1) - Ke^{-rT} N(d_2)$$

$$d_1 = \frac{\ln(S_0 / K) + (r + \sigma^2 / 2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S_0 / K) + (r - \sigma^2 / 2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

*Source: Author*

# The Black-Scholes Formula

- Similarly for a put options with the same $K$ and $T$

$$p_0 = Ke^{-rT}N(-d_2) - S_0N(-d_1)$$

*Source: Author*

- The formula also follows from the put call parity
- The function $N(x)$ denotes the standardized cumulative normal distribution, for example NOMSDIST(x) in Excel

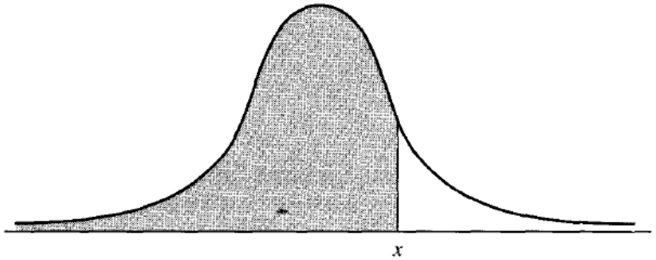



**Figure 12.3** Shaded area represents $N(x)$

Source: John Hull, Options, Futures, and Other Derivatives, 5th edition

# Derivation of the BS Formula (for a European Call Option)

Our goal is to calculate $E[\max(S-K,0)] = \int_K^\infty (S-K)g(S)dS$ with $S = S_T$

$$\ln S \sim N\left(m, w^2\right), \text{ where } m = \ln S_0 + \left(r - \frac{1}{2}\sigma^2\right)T \text{ and } w^2 = \sigma^2 T.$$

Substitute $X = \dfrac{\ln S - m}{w}$ and so $g(S)dS = \varphi(X)dX = \dfrac{1}{\sqrt{2\pi}} e^{-X^2/2} dX$

$$E[\max(S-K,0)] = \int_{(\ln K - m)/w}^{\infty} (e^{Xw+m} - K)\varphi(X)dX =$$

$$= \int_{(\ln K - m)/w}^{\infty} \frac{1}{\sqrt{2\pi}} e^{(-X^2+2Xw+2m)/2} dX - K \int_{(\ln K - m)/w}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-X^2/2} dX$$

The second integral is easy:

$$\int_{(\ln K - m)/w}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-X^2/2} dX = N(-(\ln K - m)/w) \qquad N(x) = \Phi(x) = \Pr[X \le x] = \int_{-\infty}^{x} \varphi(X)dX$$

# Derivation of the BS Formula

Regarding the first integral:

$$\frac{-X^2+2Xw+2m}{2}=\frac{-(X-w)^2+2m+w^2}{2}$$

$$\int_{(\ln K-m)/w}^{\infty}\frac{1}{\sqrt{2\pi}}e^{(-X^2+2Xw+2m)/2}dX=e^{m+w^2/2}\int_{(\ln K-m)/w}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-(X-w)^2/2}dX=$$

$$=e^{m+w^2/2}N(w-(\ln K-m)/w).$$

It is easy to check:

$$w-(\ln K-m)/w=\frac{-(\ln K-m)+w^2}{w}=\frac{-\ln K+\ln S_0+rT-\sigma^2T/2+\sigma^2T}{\sigma\sqrt{T}}=$$

$$=\frac{\ln S_0/K+(r+\sigma^2T)/2}{\sigma\sqrt{T}}=d_1,$$

$$-(\ln K-m)/w=\frac{\ln S_0/K+(r-\sigma^2T)/2}{\sigma\sqrt{T}}=d_2,\qquad e^{m+w^2/2}=e^{\ln S_0+rT}=S_0e^{rT}$$

And so $\boxed{c=e^{-rT}\left(S_0e^{rT}N(d_1)-KN(d_2)\right)=S_0N(d_1)-e^{-rT}KN(d_2)}$

Source: Author

# The Greeks

- Partial derivatives of an option (portfolio) market value measure sensitivity with respect to the relevant variables
- **Delta, Gamma** – the 1st and the 2nd derivatives w.r.t. the underlying asset price
- **Vega** – the derivative w.r.t. the volatility variable
- **Rho** – the derivative w.r.t  the interest rate
- **Theta** – the derivative w.r.t. to time, ususally measured as the change of value „per day“
- The Greeks are used for hedging… *Delta-hedging, Vega-hedging, Gamma-hedging* e.t.c

# The Greeks and Delta Hedging

For a European call option on a non-dividend paying stock
For a put option use the put-call parity $c+Ke^{-rT} = p+S_0$

$$\Delta_{\text{call}} = \frac{\partial c}{\partial S} = N(d_1)$$

$$\mathrm{V}_{\text{call}} = \mathrm{V}_{\text{put}} = S\sqrt{T-t}\,N'(d_1)$$

$$\Delta_{\text{put}} = \frac{\partial p}{\partial S} = N(d_1) - 1$$

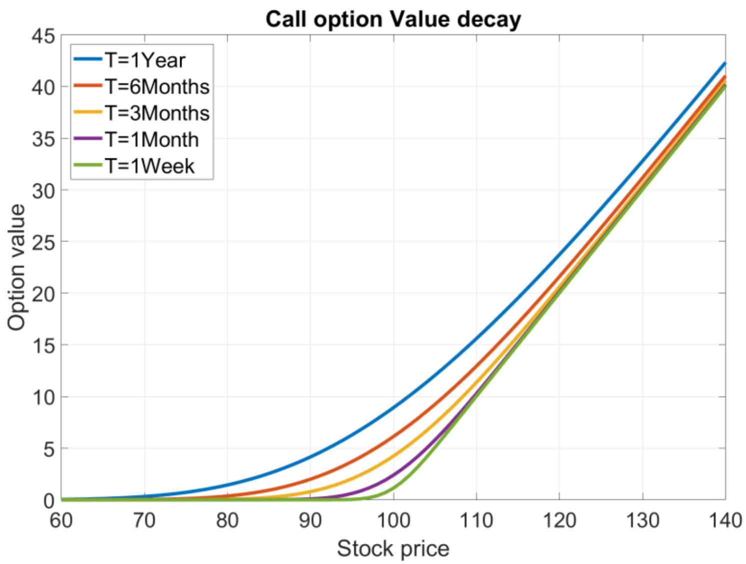
$$\rho_{\text{call}} = K(T-t)e^{-r(T-t)}N(d_2)$$

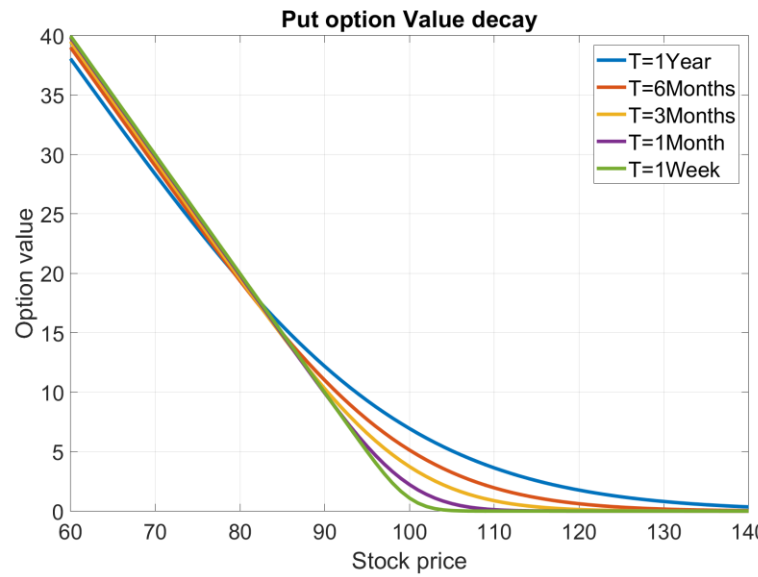
$$\Gamma_{\text{call}} = \Gamma_{\text{put}} = \frac{N'(d_1)}{S\sigma\sqrt{T-t}}$$

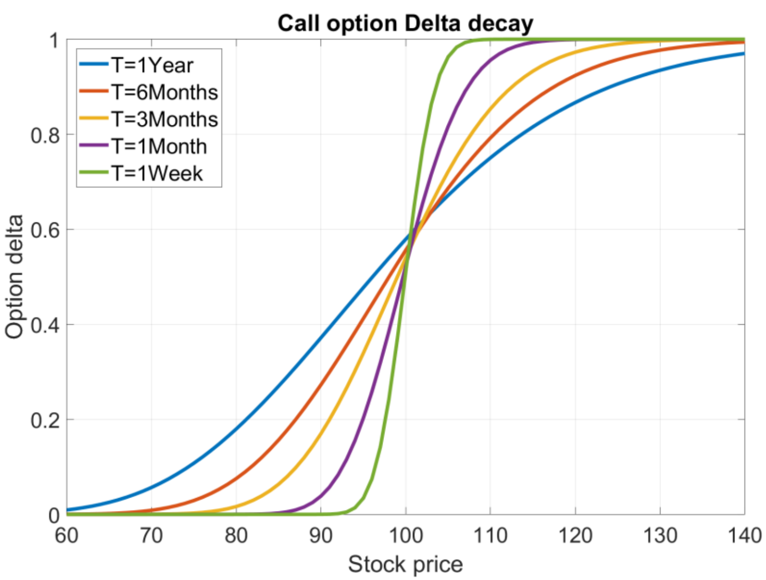
$$\Theta_{\text{call}} = -\frac{SN'(d_1)\sigma}{2\sqrt{T-t}} - rKe^{-r(T-t)}N(d_2)$$

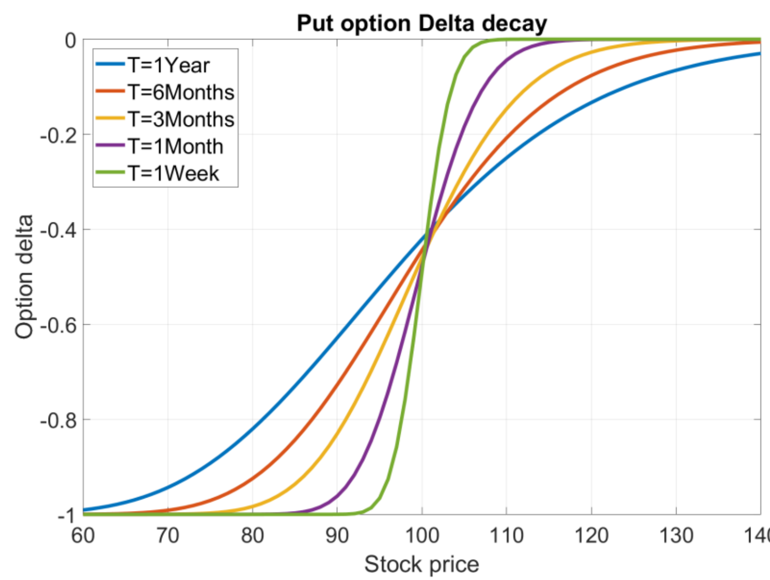
**Example (Delta hedging):** Consider a portfolio consisting of three different option positions on the same underlying asset with deltas +100, -60, +30. Propose an appropriate delta hedging using a forward contract with the same (no income) underlying.

# Value and Delta decay



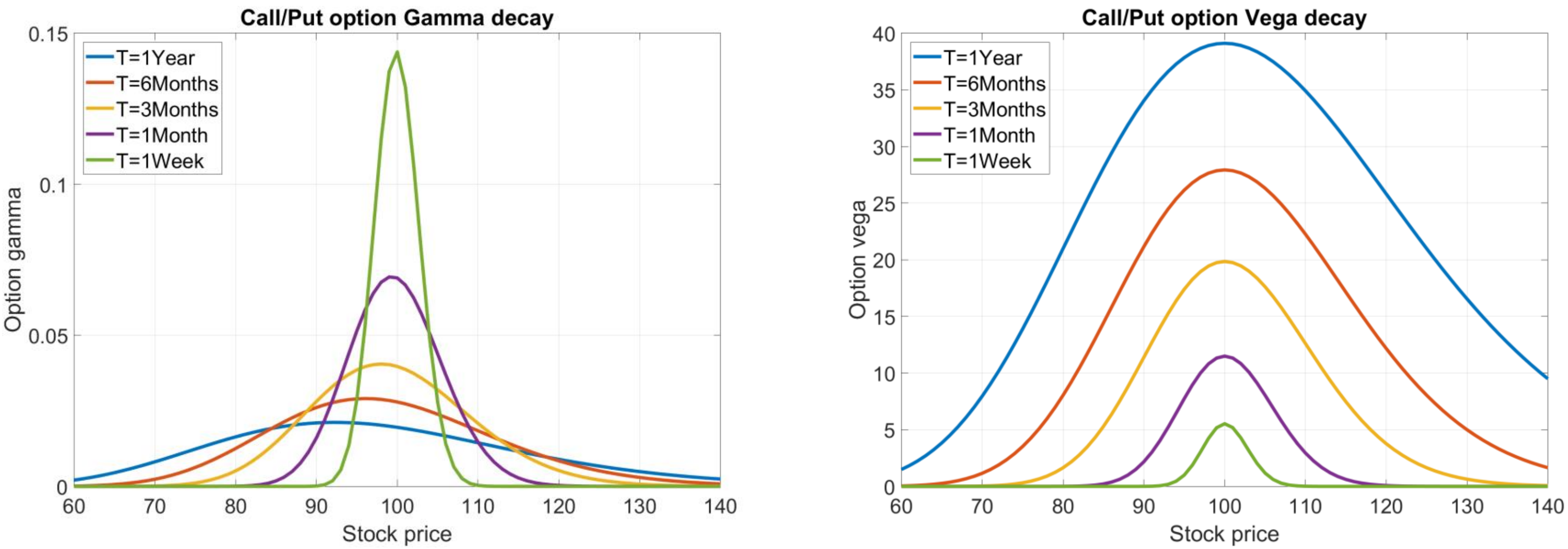




*Source: MATLAB*

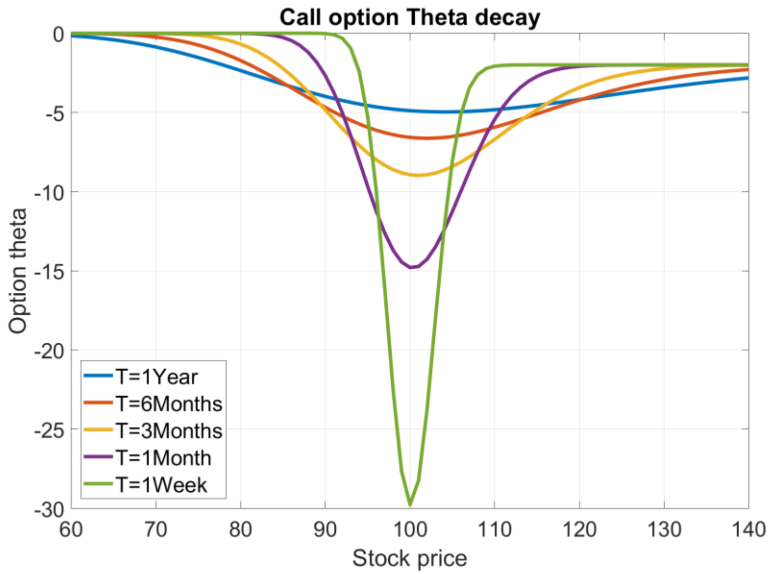
# Gamma and Vega decay

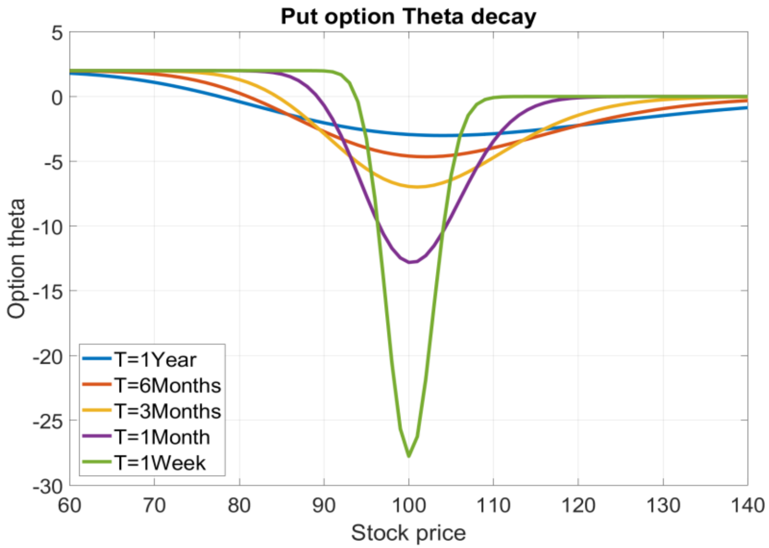


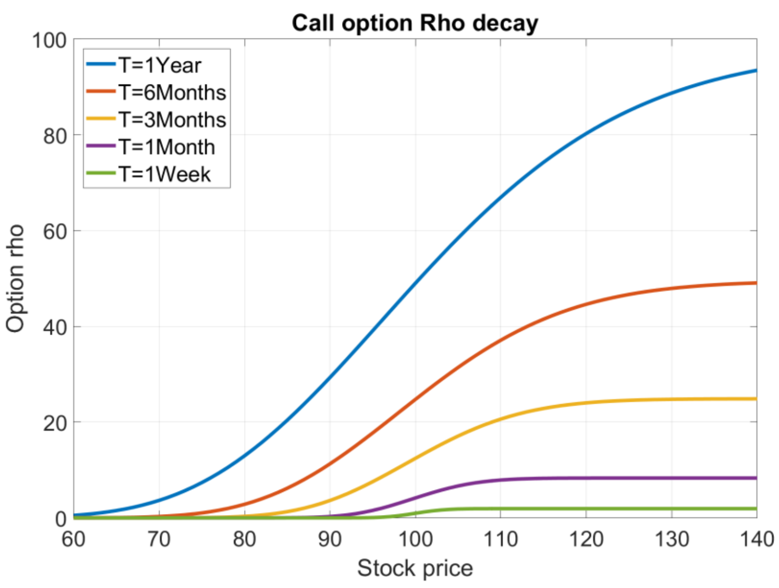

*Source: MATLAB*

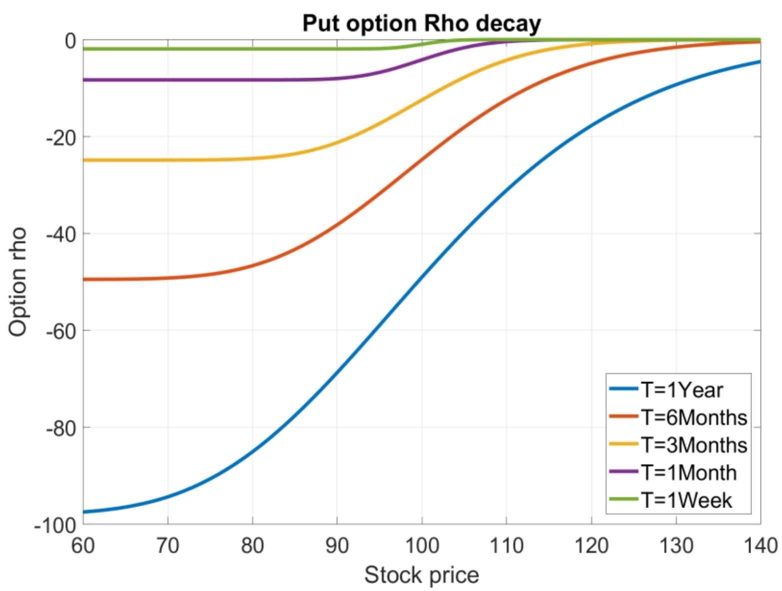
- Gamma and Theta are the same for Call and Put options on the same underlying and with the same strike price and maturity
- All charts were generated for K=100, σ=20% and r=2%

# Theta and Rho decay




*Source: MATLAB*

# Call vs. put option theta

- We can see that while the call option theta is always negative (i.e. option value decreases with time), put option theta is positive for deeply in-the money options
- Time to maturity influences the option price through the following channels:
  - Discounting of future payoff (negative for call and put options)
  - Drift rate of the stock price (positive for call, negative for put)
  - Through the volatility of future payofff (positive for both options)
- The drift rate channel is always stronger than the discounting channel for call options. Together with the volatility channel it thus assures that option value decreases as T decreases.
- For put options the effects of the drift and discounting work in the opposite direction (with respect to T) than the volatility effect
- The result is that for deeply in the money call options (for which vega is low), the value may increase as T decreases
- This is why American put options on a non-dividend paying stock may sometimes be exercised before maturity (unlike calls)

# Delta, Gamma, Vega hedging

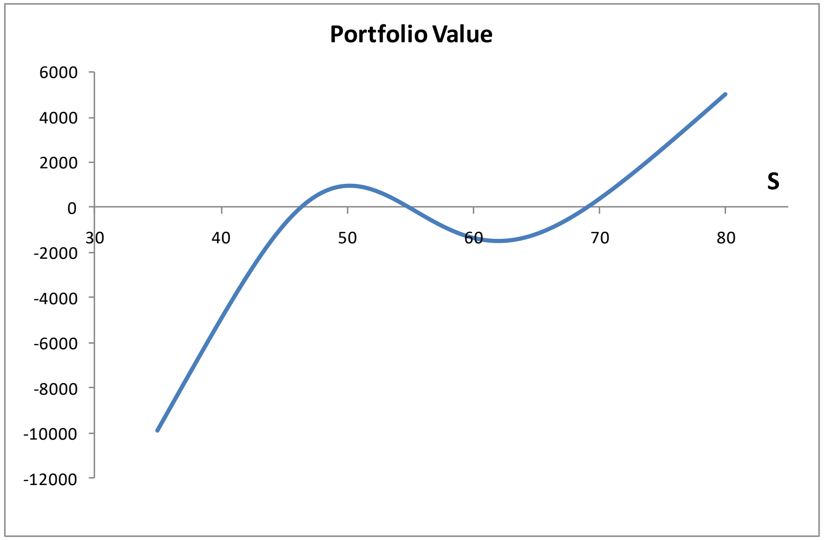
- **Delta hedging** – With stock or futures
- **Gamma hedging** – With another option
- **Theta hedging** – With another option (by hedging Gamma, we hedge Theta as well)
- **Vega hedging** – With another option (if we want to hedge Gamma as well, we need two options)
- **Rho hedging** – With money market instruments

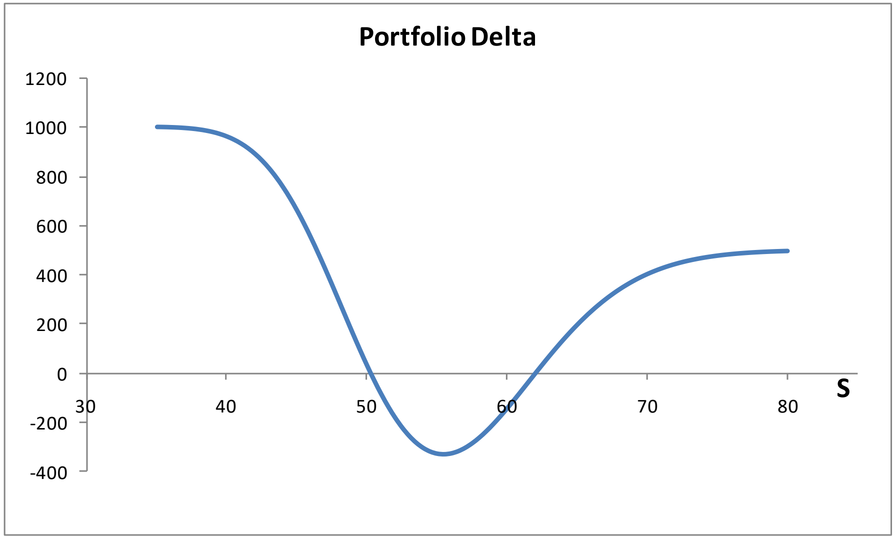
- Hedging all Greeks: First hedge Vega and Gamma by using 2 other options (with short and long maturity), then hedge the remaining Delta with stock or futures, and finally hedge Rho on the money market

# Delta Hedging Example

- A trader that has just sold an at-the-money straddle on 1000, and bought an out-of the money call on 1500 non dividend paying stocks.
- The actual stock price is $S = 50$ , we assume constant interest rate $r = 1\%$ and volatility $\sigma = 15\%$ . All the three European options have six months to maturity, $T = 0.5$ , the strike price of the straddle call and put options is $K = 50$ , and the strike of the out-of-the money call is $K = 60$ . The trader has received a net initial premium of €5 000 and currently is in a profit around €940.

# Delta Hedging Example




*Source: Author*

Development of the option portfolio value and delta depending on the underlying stock

# Delta Hedging Example

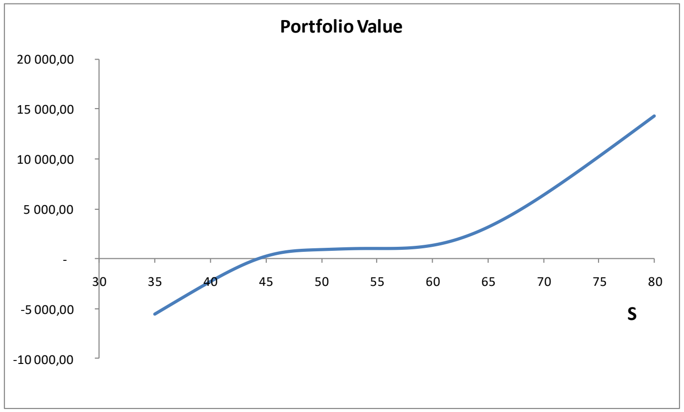
| Day | S | Port. delta | Or. port. value | Delta pos. | Buy/sell | Cost | Cum.cost | Total portf. |
|---|---|---|---|---|---|---|---|---|
| 1 | 50,00 | 29,50 | 942,89 | -29 | -29 | 1 450,00 | 1 450,00 | 942,89 |
| 2 | 51,00 | -85,15 | 913,89 | 85 | 114 | -5 814,00 | -4 364,00 | 884,89 |
| 3 | 51,50 | -135,87 | 858,71 | 135 | 50 | -2 575,00 | -6 939,00 | 872,21 |
| 4 | 52,00 | -181,37 | 779,57 | 181 | 46 | -2 392,00 | -9 331,00 | 860,57 |
| 5 | 53,00 | -254,95 | 560,49 | 254 | 73 | -3 869,00 | -13 200,00 | 822,49 |
| 6 | 53,50 | -282,43 | 426,55 | 282 | 28 | -1 498,00 | -14 698,00 | 815,55 |
| 7 | 54,00 | -303,51 | 280,53 | 303 | 21 | -1 134,00 | -15 832,00 | 810,53 |
| 8 | 55,00 | -326,54 | -35,01 | 326 | 23 | -1 265,00 | -17 097,00 | 797,99 |
| 9 | 57,00 | -302,30 | -675,10 | 302 | -24 | 1 368,00 | -15 729,00 | 809,90 |
| 10 | 59,00 | -206,34 | -1 190,40 | 206 | -96 | 5 664,00 | -10 065,00 | 898,60 |

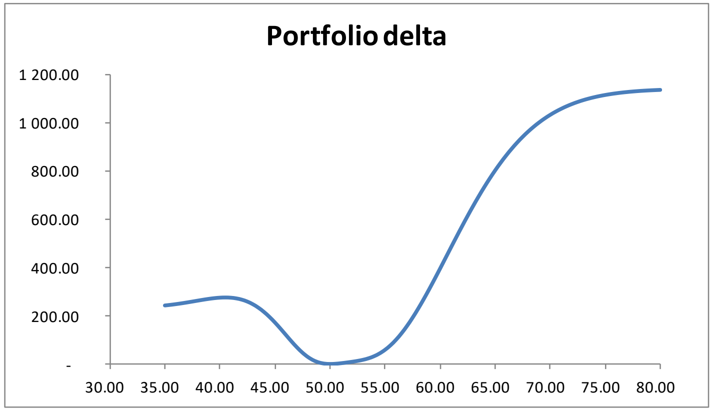
*Source: Author*

Simulation of a dynamic portfolio delta-hedging

# Gamma and Vega Hedging Example

- Gamma of the portfolio -122 can be offset by a long position in more liquid options, e.g. by a long one month straddle




*Source: Author*

- In order to hedge the Vega (-191 mil. for the Gamma-hedged portfolio) we also need to use longer maturity options, e.g. 1 year, and solve a set of equations with two unknowns

# Higher order Greeks

- Black-Scholes model uses only one second-order Greek (gamma), but more complex option pricing methods (Heston model, Vanna-Volga method, etc.) work commonly with second-order Greeks:

$$Vanna = \frac{\partial \Delta}{\partial \sigma} = \frac{\partial \nu}{\partial S} = \frac{\partial^2 f}{\partial S \partial \sigma} \qquad Gamma = \frac{\partial \Delta}{\partial S} = \frac{\partial^2 f}{\partial S^2}$$

$$Vomma = \frac{\partial \nu}{\partial \sigma} = \frac{\partial^2 f}{\partial \sigma^2} \qquad Speed = \frac{\partial \Gamma}{\partial S} = \frac{\partial^3 f}{\partial S^3}$$

- Large option traders and market-makers also commonly use second or even third-order greeks (e.g. Speed) to hedge their portfolios

# Options on Income Paying Assets

- If an asset with price $S(t)$ pays continuous income $q$ than its drift must be reduced by $q$ compared to assets paying no income and with the same risk
- In the risk neutral world $dS = (r\text{-}q)Sdt + \sigma Sdz$
- If $U(t) = S(t)e^{-q(T-t)}$ then $dU = rUdt + \sigma Udz$ and $S(T) = U(T)$, $U(0)=S_0e^{-qT}$ … reinvestment policy
- ➢ Use the same formulas (put-call parity, pricing, and Greeks) as for options on non-dividend stocks but replace $S_0$ by $S_0e^{-qT}$ !!!
- Applicable to options on **stock indices or on foreign currencies** ($q = r_{\text{foreign}}$)

# Options on Futures

- Options on futures are settled by entering in a futures position and by an immediate gain-loss settlement (i.e. $F$-$K$ or $K$-$F$)
- For a futures on a non-dividend paying stock

$$F = Se^{r(T-t)}$$

- Consequently in the risk neutral world

$$dF = (r-r)Fdt + \sigma Fdz = \sigma Fdz$$

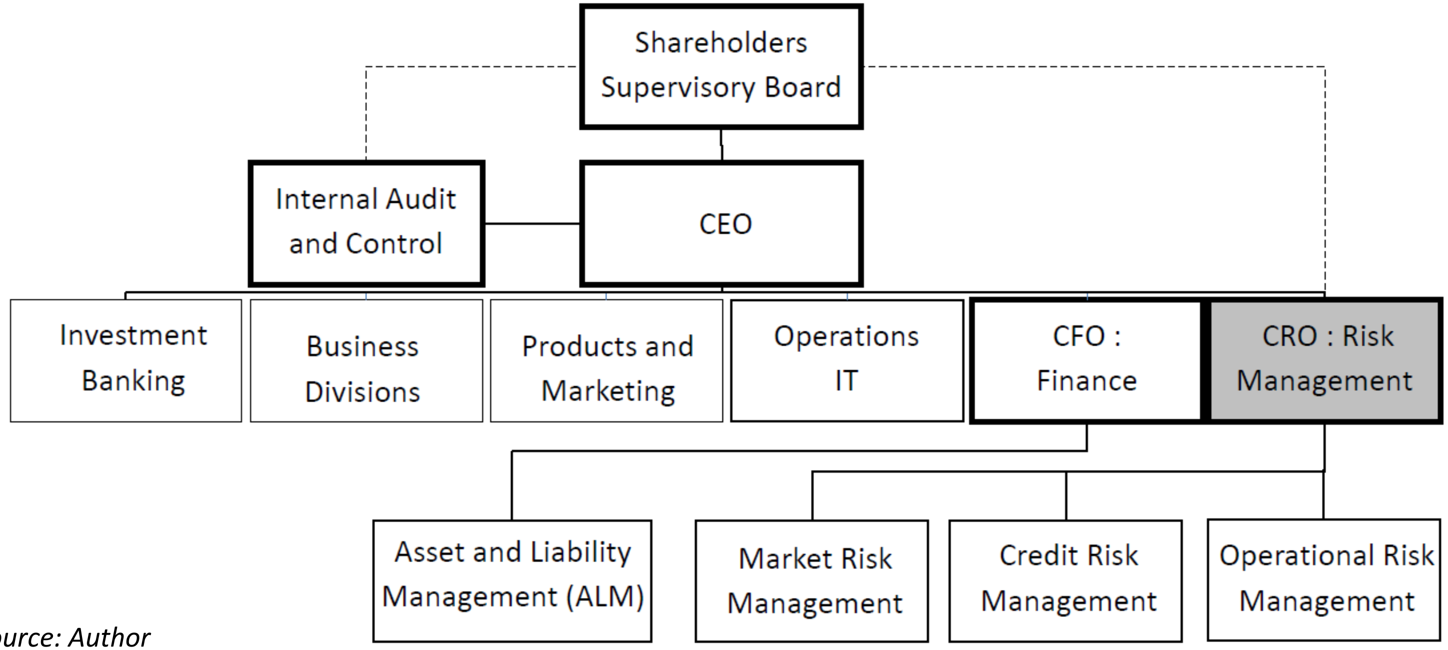
- ➢ Use the same formula as for options on non-dividend stocks but replace $S_0 = F_0$ by $F_0 e^{-rT}$ $(q = r)$
- Options on futures are usually American type – the formula above applies only to European options on futures! … European and American prices coincide for calls not for puts

# Content

- ✓ Introduction – overview of B.-S. option pricing and hedging
- ➢ Market Risk Management
- Estimating volatilities and correlations
- <u>Interest Rate Derivatives Pricing-Martingale and measures</u>
- <u>Standard Market Model</u>

# Market Risk Management and Measurements

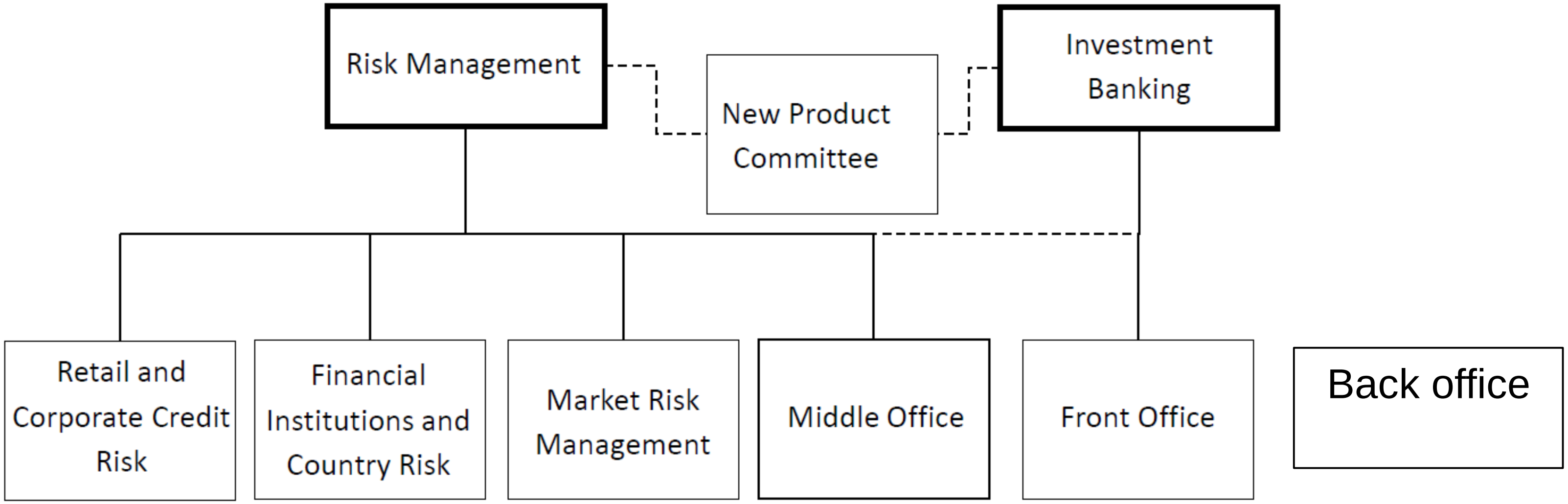
- Risk: market, credit, operational, liquidity
- Independence of risk management organization is of key importance




*Source: Author*

# Market Risk Management

- Market Risk Department sets limits, measures, and controls not only the market itself but also the counterparty risk
- A typical organization structure between the Risk and Investment Banking




*Source: Author*

# Classical Market Risk Measures

- Value of investment – number of securities held
- Treatment of unsettled spots/forwards
- How to combine positions in different securities – take into account correlations?
- The simple measures are often used to set basic limits – not precise, but easy to evaluate

# Interest Rate Risk Management

- Gap analysis, Duration and Basis Point Value

$$BPV = V(r - 0.01\%) - V(r)$$

$$BPV \cong D \times V \times 0.01\%$$

- Can be also used for a set of maturities

$$BPV^i = V(r_1, ..., r_i - 0.01\%, ..., r_n) - V(r_1, ..., r_n)$$

$$\Delta V \cong \sum_{i=1}^{m} x_i BPV^i$$

- Portfolio of bonds is divided into cash-flows for each time from 1…n
- Sensitivities to 1bp parallel shift, twist of the term structure - Principal Component Analysis (PCA)

# General Delta Sensitivity Approach

- Sensitivities with respect to relevant market factors

$$\Delta V = V\left(x_1 + \Delta x_1, \ldots, x_m + \Delta x_m\right) - V\left(x_1, \ldots, x_m\right) \cong \sum_{i=1}^{m} \frac{\partial V}{\partial x_i} \Delta x_i$$

- Or including the second order derivatives

$$\Delta V \cong \sum_{i=1}^{m} \frac{\partial V}{\partial x_i} \Delta x_i + \frac{1}{2} \sum_{i,j=1}^{m} \frac{\partial^2 V}{\partial x_i \partial x_j} \Delta x_i \Delta x_j$$

- The risk factors can be: Stock prices, stock indices, interest rates, yields, exchange rates, commodity prices, implied volatilities, etc.

# Value at Risk

- "How large loss can we suffer on a portfolio (or business activity)? And what is the probability of such a loss?"

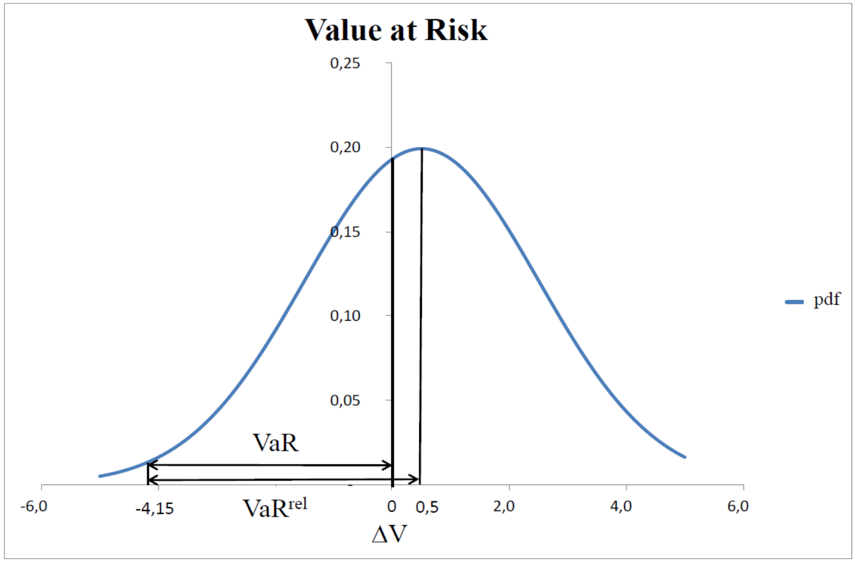
$$\Delta V = V\left(\mathbf{x}(t+\Delta t)\right) - V\left(\mathbf{x}(t)\right)$$

$$VaR^{abs}(\Delta t, \alpha) = -q_{1-\alpha} \qquad VaR^{rel}(\Delta t, \alpha) = E[\Delta V] - q_{1-\alpha}$$

CVaR – expected shortfall

$$CVaR^{abs} = -E[\Delta V | \Delta V \leq -VaR^{abs}]$$

$$CVaR^{rel} = E[\Delta V] - E[\Delta V | \Delta V \leq -VaR^{abs}]$$




*Source: Author*

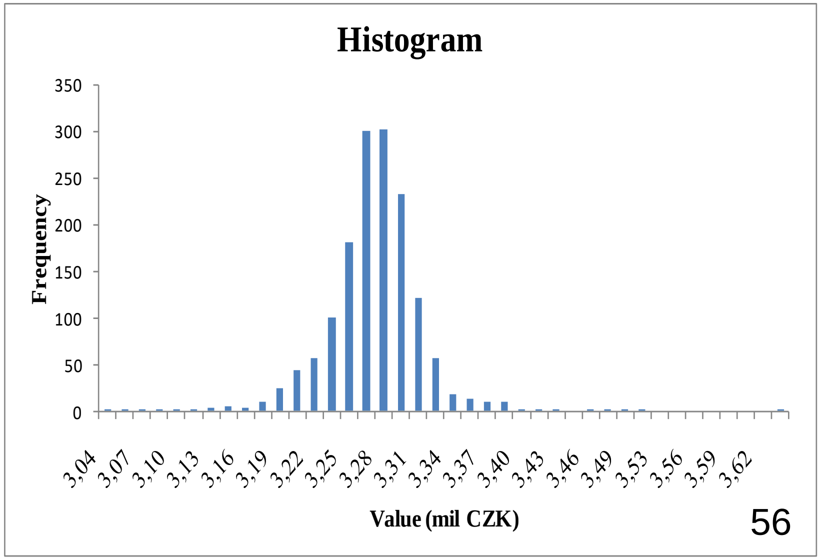
# VaR estimation methods

- Historical simulation
- Analytical VaR
  - Linear vs. Non-linear approximations
  - Gaussian vs. t-distribution
- Monte carlo simulations
  - Simulating returns from a distribution
  - Simulation of stochastic processes (SVJD)
- Advanced approaches (extreme value theory, copula functions, etc.)

# Historical VaR

- For all factors $j = 1, \dots, m$ and times $j = 1, \dots, N$ calculate the historical returns: $r_j^i = \frac{x_j^i - x_j^{i-1}}{x_j^{i-1}}$
- Compute projected portfolio returns for all scenarios: $y_j^i = x_j(t)\left(1 + r_j^i\right)$
- The scenarios $\left\{V\left(\mathbf{y}^1\right), \dots, V\left(\mathbf{y}^N\right)\right\}$ represent the empirical multivariate distribution of $V(\boldsymbol{x}(t + \Delta t))$
- Empirical quantiles can be used to compute VaR and CVaR
- Standard error: $s = \frac{1}{f(q)}\sqrt{\frac{(1-\alpha)\alpha}{N}}$
- Alternativelly we can get $s$ with bootstrapping

*Source: Author*



# Analytical VaR

- In the case of **normally distributed** portfolio returns VaR and CVaR depend only on volatility σ:

$$VaR^{rel} = \mu - \left(\mu + \sigma \cdot q_{1-\alpha}^{N}\right) = -\sigma \cdot q_{1-\alpha}^{N} = \sigma \cdot q_{\alpha}^{N} \qquad CVaR = \frac{N'(q_{\alpha}^{N})}{1-\alpha}\sigma$$

- We can use **linear aproximation** to estimate portfolio variance as:

$$\Delta V \cong \sum_{j=1}^{m} \alpha_j r_j \qquad \sigma_V^2 = \boldsymbol{\alpha}' \cdot \mathbf{Cov} \cdot \boldsymbol{\alpha} = \sum_{i,j=1}^{m} \alpha_i \alpha_j \operatorname{cov}_{i\,j}$$

- Where $r_j$ denotes the factor returns, $a_j$ the sensitivity of the portfolio to the factors, and $Cov_{i,j}$ is the covariance matric of factor returns
- Change of time horizon: $VaR(n\Delta t, \alpha) = \sqrt{n}\, VaR(\Delta t, \alpha)$

# Analytical VaR - Example

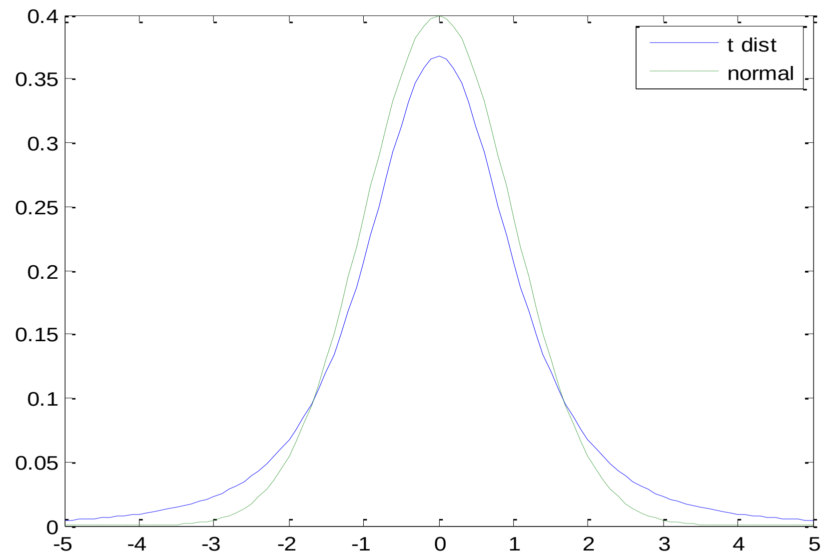
- Consider a Czech investor holding a portfolio of:
  - $a$ units of US Stock S&P500 ETF with price $X_1$ in USD
  - $b$ units of EuroStoxx 600 ETF with price $X_2$ in EUR
- And denote $S_1$ the USD/CZK rate and $S_2$ the EUR/CZK rate
- The value of the portfolio in CZK is: $V = aS_1X_1 + bS_2X_2$
- We can use linear aproximation to aproximate the change of the portfolio value:
- $\Delta V \cong \sum_{i=1}^{4} \frac{\partial V}{\partial x_i} \Delta x_i = \sum_{i=1}^{4} a_i r_i$
- Where: $a_i = \frac{\partial V}{\partial x_i} x_i$ and $r_i \frac{\Delta x_i}{x_i}$
- Portfolio variance is: $\sigma_V^2 = \boldsymbol{\alpha}' \boldsymbol{Cov}\, \boldsymbol{\alpha} = \sum_{i,j=1}^{4} a_i a_j Cov_{i,j}$
- And portfolio VaR: $VaR^{Rel} = \sigma_V q_p^N$

# Cornish-Fisher expansion

- Better approximation of $\Delta V$ can be achieved by including the second order terms:
- $\Delta V \cong \sum_{j=1}^{m} \alpha_j r_j + \frac{1}{2} \sum_{i,j=1}^{m} \gamma_{ij} r_i r_j$
- Where $\gamma_{ij} = \frac{\partial^2 V}{\partial x_i x_j} x_i x_j$
- Quantiles of $\Delta V$ can then be estimated with the **Cornish-Fisher expansion**:
- $q_p[\Delta V] \cong \mu_V + w_p \sigma_V$
- Where $w_p = q_p^N + \frac{1}{6}\left[\left(q_p^N\right)^2 + 1\right] \xi_V$
- Where $\xi_V$ is the skewness of $\Delta V$

# Alternative parametric distributions

- Alternative parametric distributions – e.g. **Student t-distribution**
- Aproximates the fat tails of the return distribution
- Variable $\sqrt{\frac{v}{v-2}}\frac{X-\mu}{\sigma}$ has a *t*-distribution with $v$ degrees of freedom
- VaR is estimated by using quantiles of the *t*-distribution




*Source: MATLAB, Author*

# *t*-Distribution VaR - Example

- Lets consider portfolio $V$ with parameters $\mu = 0.5$ and $\sigma = 2$
- Table below shows the values of VaR and CVaR with *p=95% and p=99%* for different $\nu$
- With infinite degrees of freedom the *t*-distribution converges to the normal distribution

| Degrees of freedom (v) | VaR(95%) | CVaR(95%) | VaR(99%) | CVaR(99%) |
|---|---|---|---|---|
| **3** | 2.71 | 4.47 | 5.24 | 8.11 |
| **5** | 3.11 | 4.54 | 5.21 | 6.88 |
| **7** | 3.24 | 4.2 | 4.94 | 5.99 |
| **Infinite (normal)** | 3.29 | 4.12 | 4.65 | 5.33 |

*Source: Author*

# Disadvantages of VaR

- Disadvantages of VaR: model dependence, volatility and correlations estimates dependence (CHF example)


*Source: Barchart*

# Basel Capital Accord

- BCBS 1988: Basel I or "The Accord"
- BCBS 1996: Market Risk Amendment
- BCBS 2004: Basel II
- BCBS 2010: Basel III
- Regulatory capital= Credit + Market + Operational risk capital
- Market risk: standardized or internal model based approach

$$\text{Regulatory capital} = k \cdot VaR(10 \text{ days}, 99) + \text{SRC} \qquad k \geq 3$$

- Basel III introduces stressed VaR, Incremental Risk Charge (IRC), and CVA VaR
- Fundamental Review of the Trading Book (2012) – Basel Consultative document proposing to use CVaR

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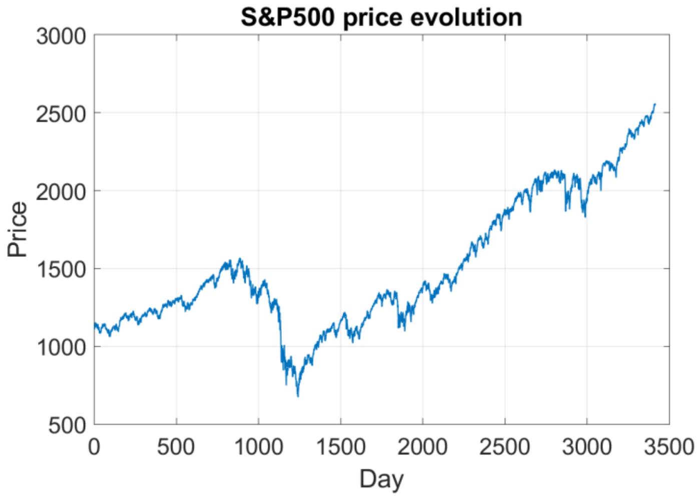
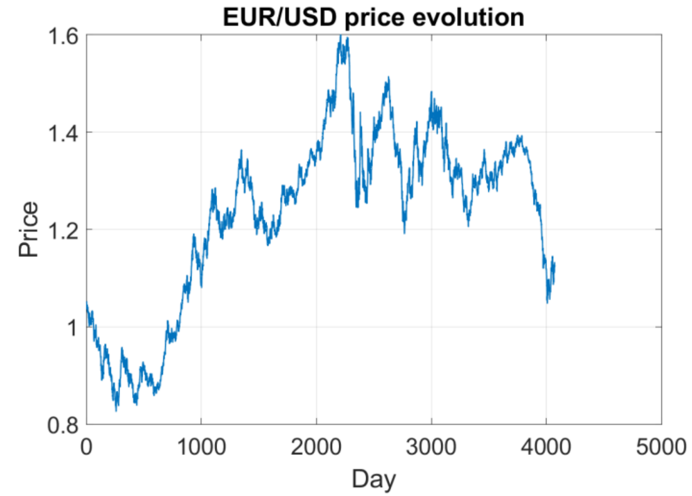
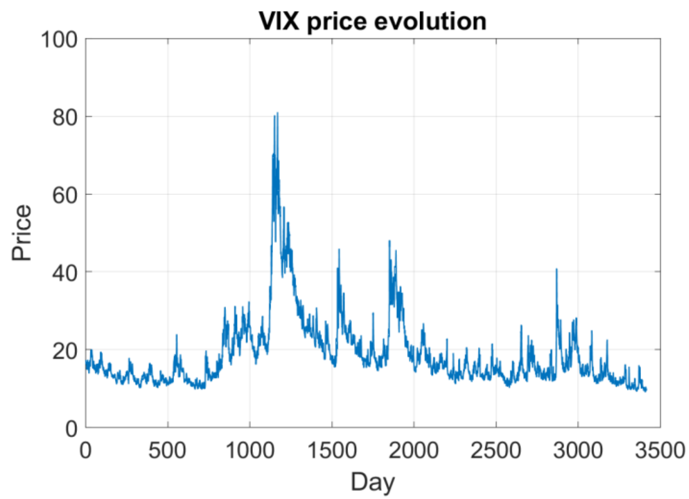
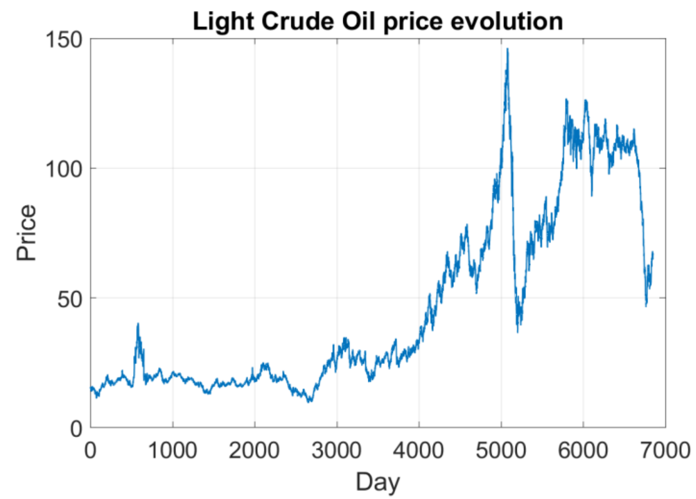
# Asset price movements



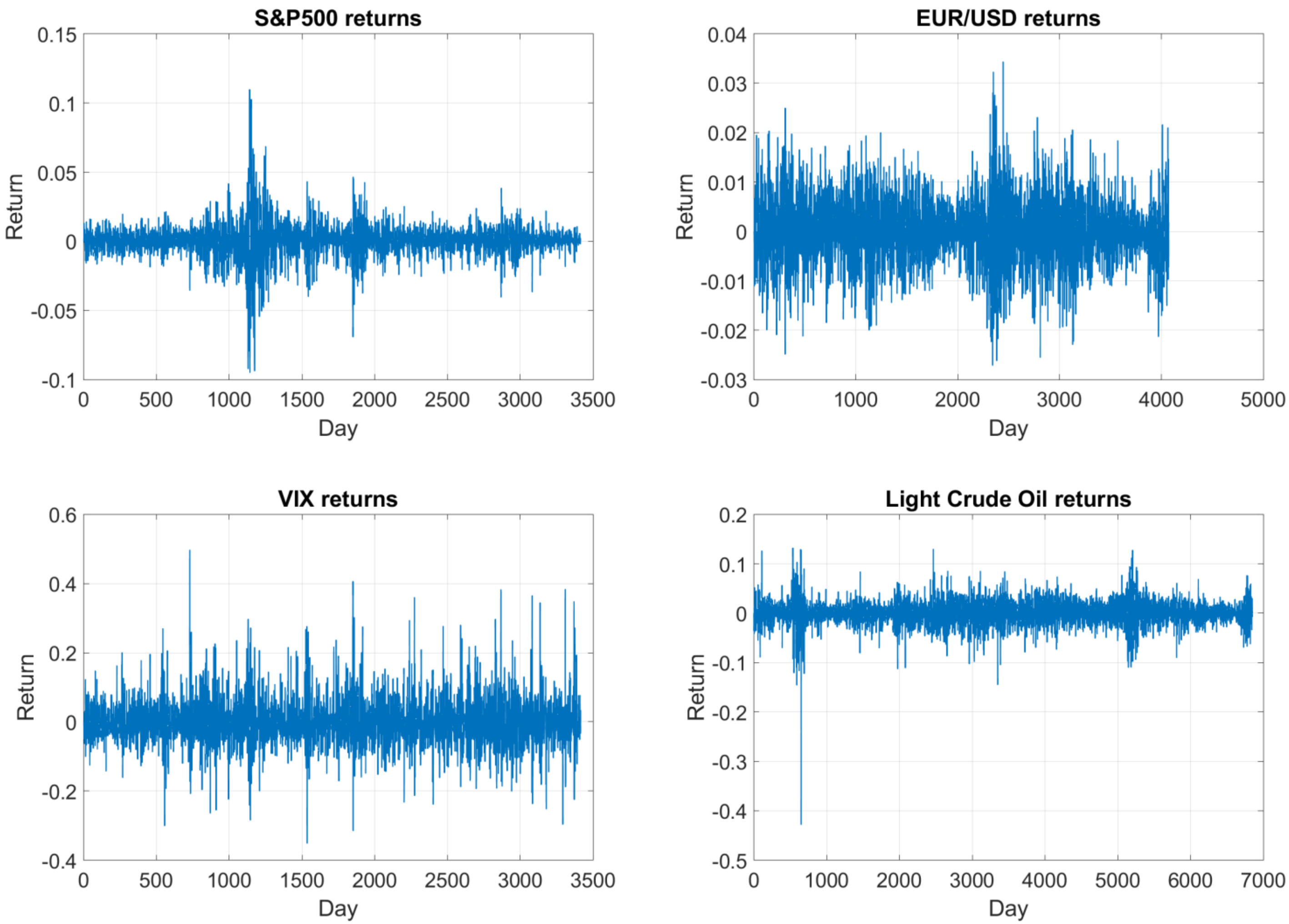



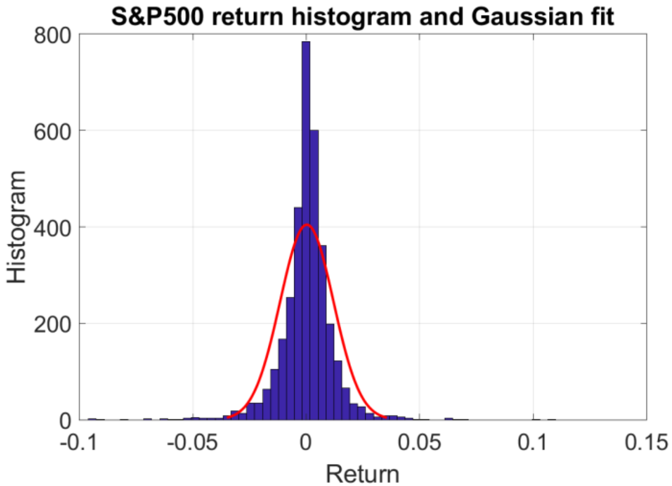

*Source: MATLAB*

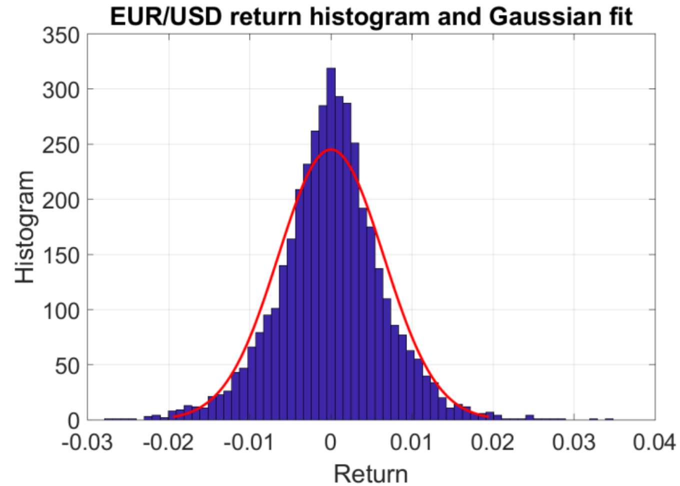
# Asset price returns

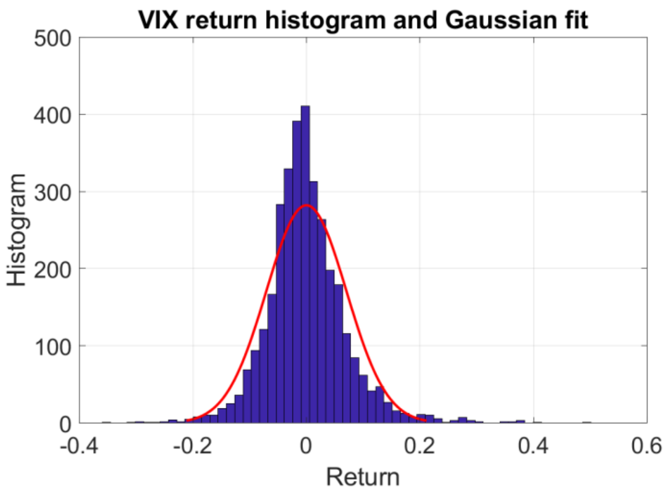


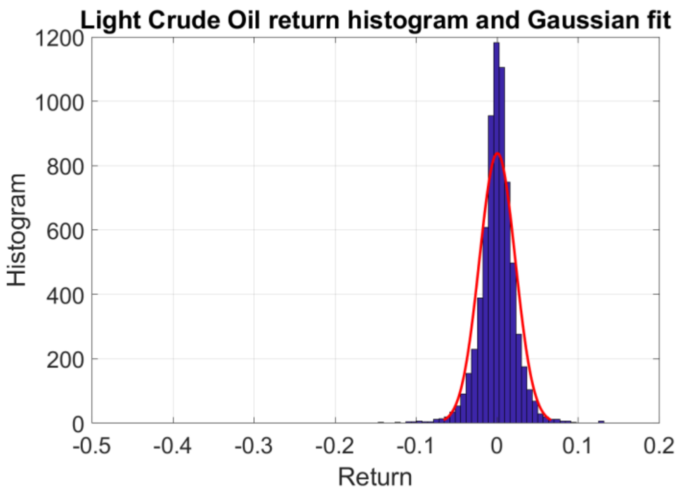

*Source: MATLAB*

# Asset return distribution



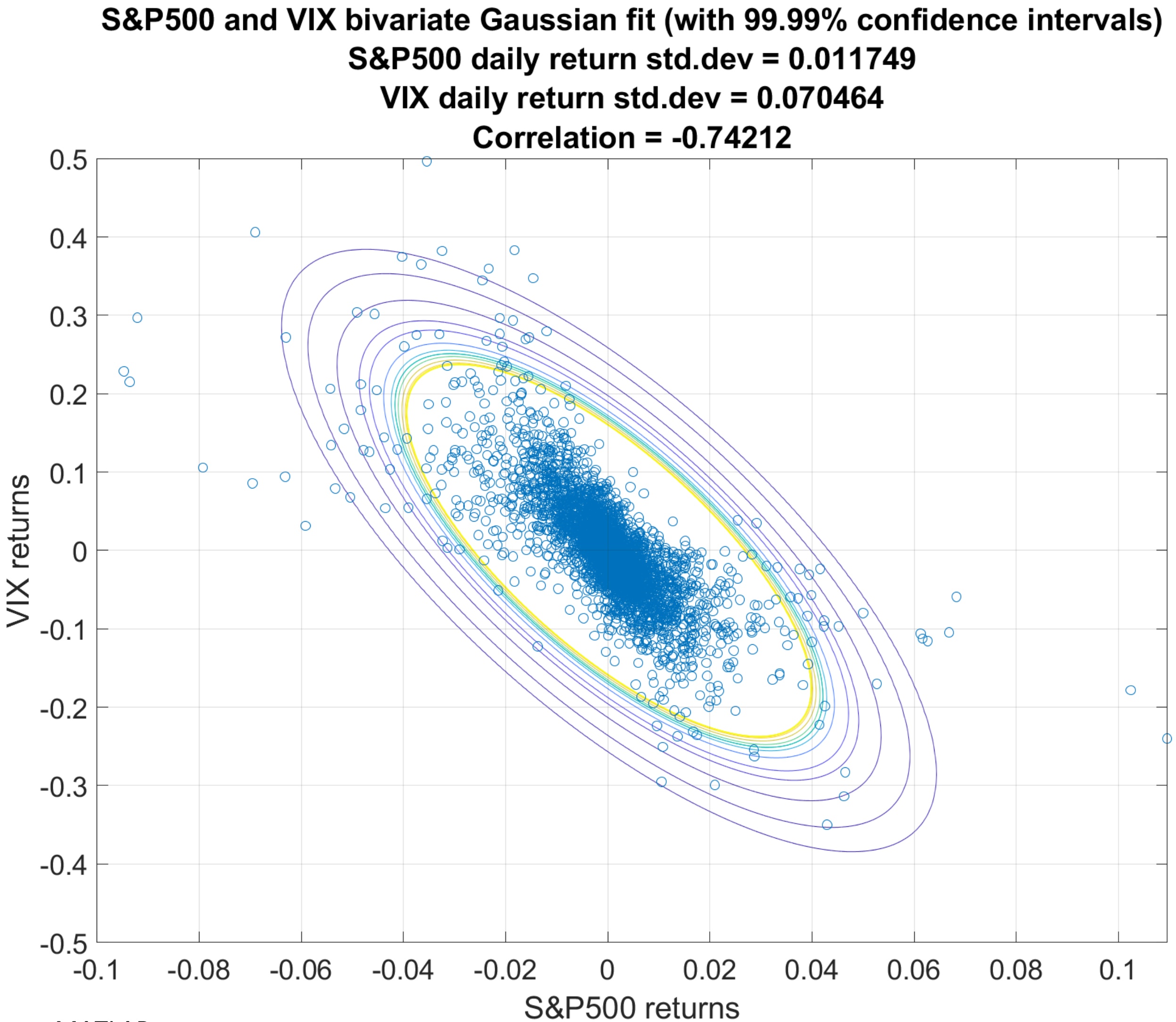




*Source: MATLAB*

# Asset price correlation




*Source: MATLAB*

# Estimating Volatilities (and Correlations)

- Volatility estimated from historical data … forward looking volatilities … option pricing
- Standard approach

$$\sigma_n^2 = \frac{1}{m-1}\sum_{i=1}^{m}(u_{n-i} - \bar{u})^2 \qquad u_i = \ln\frac{S_i}{S_{i-1}}$$

- Weighted estimates

$$\sigma_n^2 = \sum_{i=1}^{m}\alpha_i u_{n-i}^2 \qquad \sum_{i=1}^{m}\alpha_i = 1$$

- EWMA – Exponentially Weighted Moving Average

$$\sigma_n^2 = \lambda\sigma_{n-1}^2 + (1-\lambda)u_{n-1}^2 \qquad \sigma_n^2 = (1-\lambda)\sum_{i=1}^{m}\lambda^{i-1}u_{n-i}^2$$

# ARCH and GARCH

*(Generalized) Auto-Regressive Conditional Heteroskedasticity*

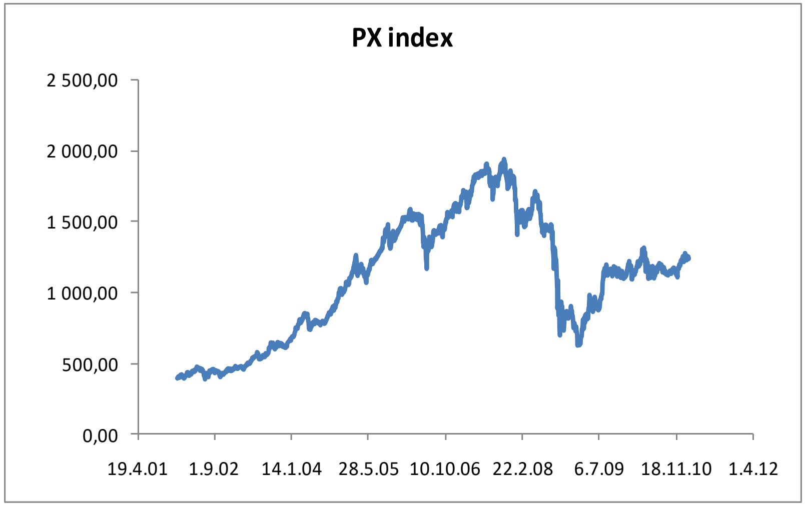
- ARCH(m) Model $\sigma_n^2 = \gamma V_L + \sum_{i=1}^{m} \alpha_i u_{n-i}^2 \qquad \gamma + \sum_{i=1}^{m} \alpha_i = 1$
- GARCH(1,1) Model $\sigma_n^2 = \gamma V_L + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2$

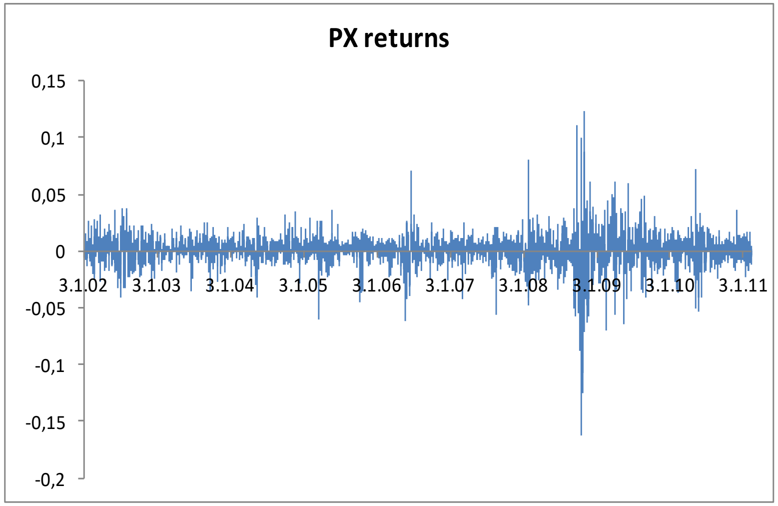
  $\gamma + \alpha + \beta = 1$
- It can be shown to correspond to the variance mean-reversion process

$$dV = a(V_L - V)dt + \xi V dz, \;\; a = \gamma, \; \xi = \alpha\sqrt{2}$$

# Estimating Volatilities Example

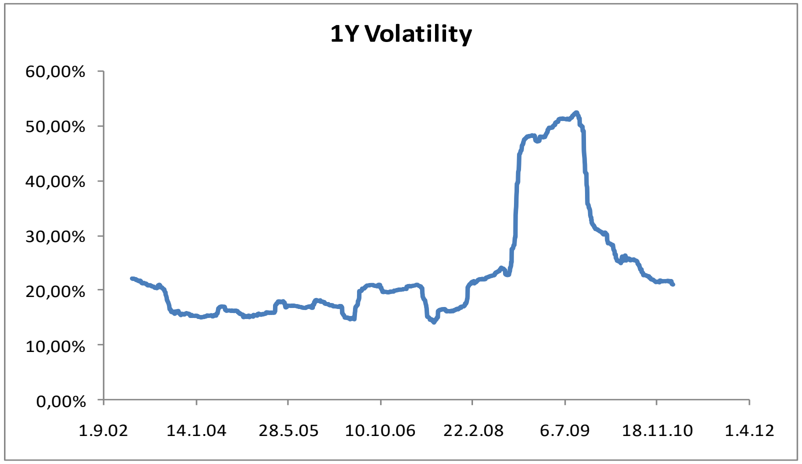
The series of PX index values and daily returns




Last 252 days, equally weighted, annualized:

$$\hat{\sigma} = 1.32\% \times \sqrt{252} = 21.01\%$$

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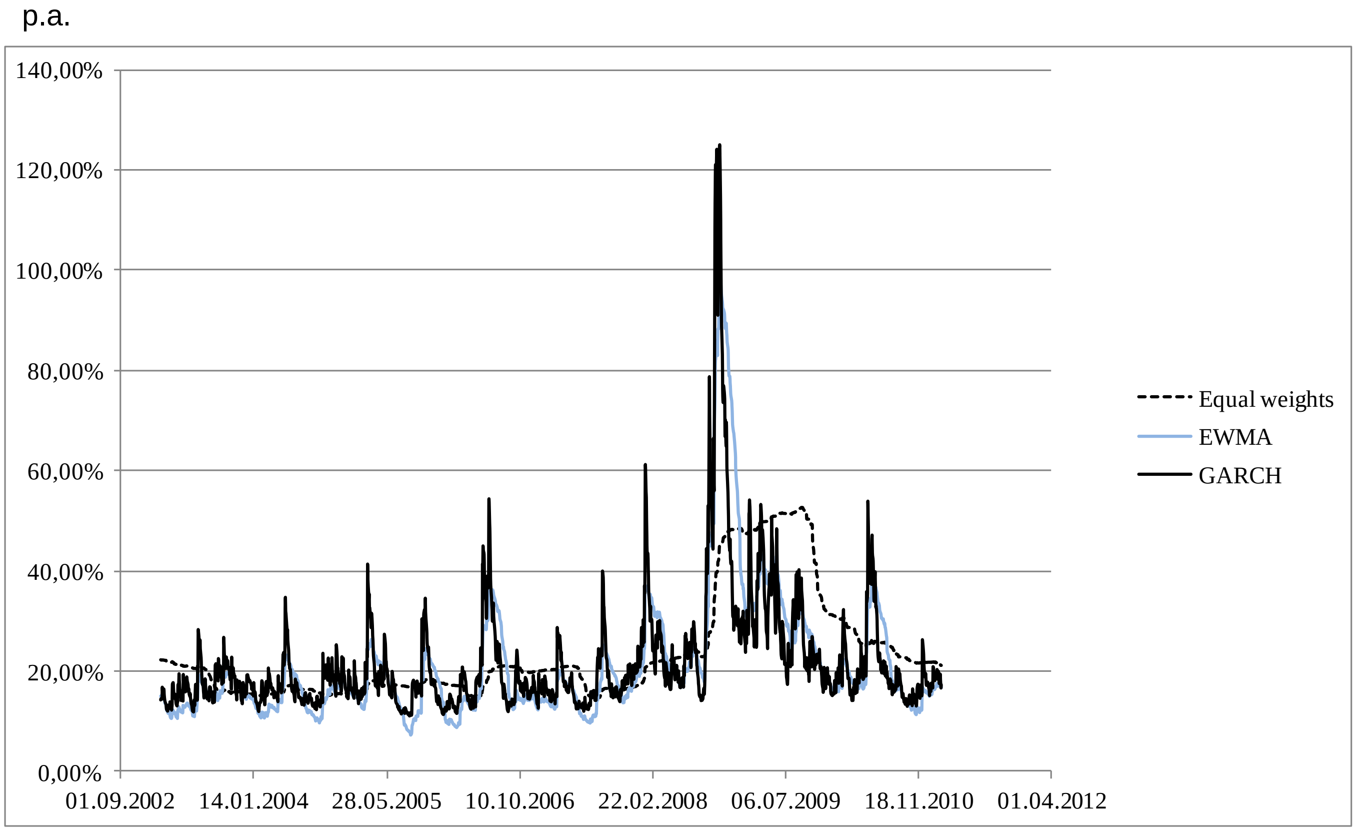
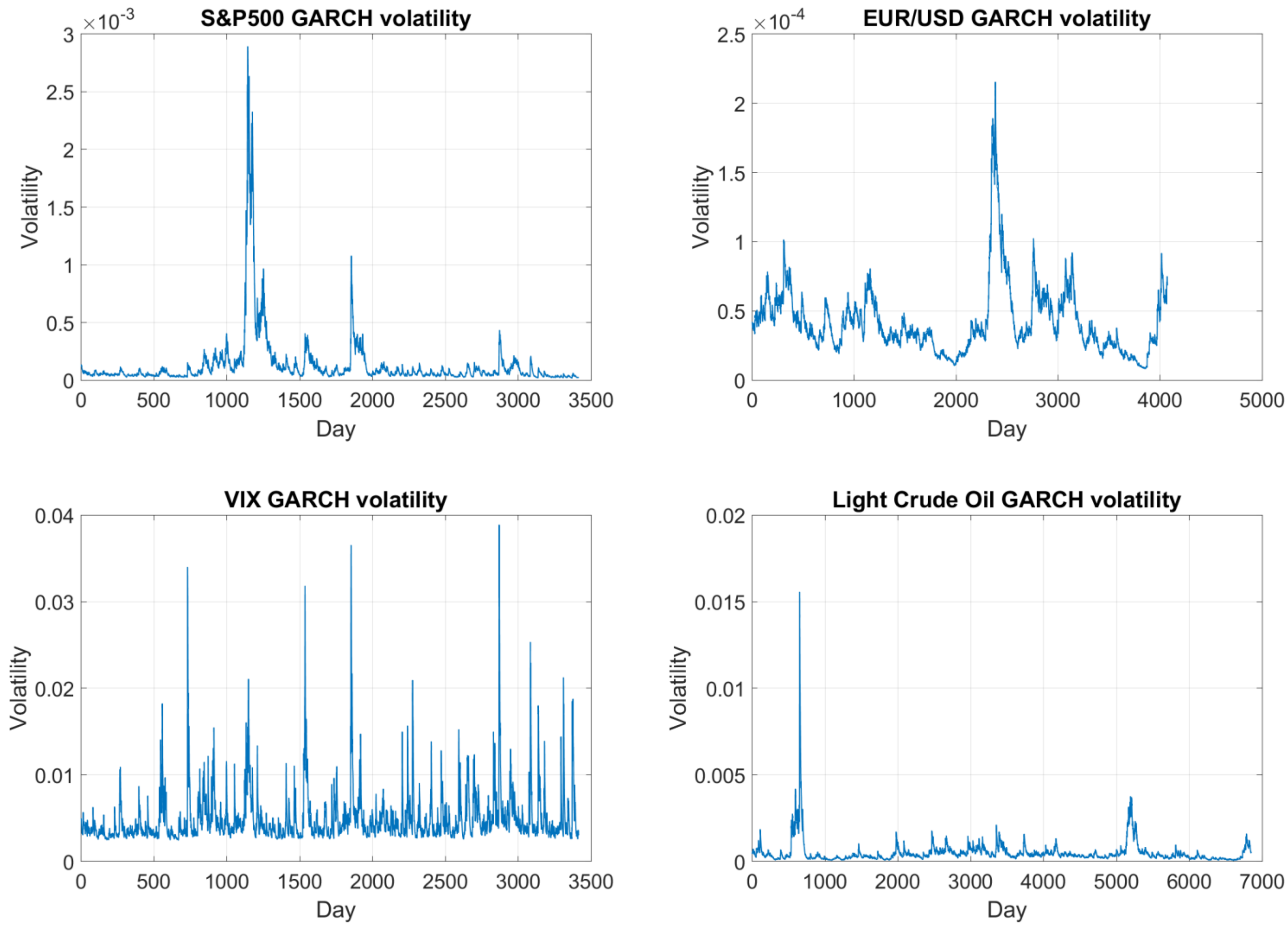
# Estimating Volatilities Example




Comparison of equal weighted, EWMA, and GARCH(1,1) historical daily PX index volatility ( $\lambda = 0.97$ ),

*Source: OxEdit*

# GARCH volatility illustration




*Source: MATLAB*

# Maximum Likelihood Calibration

- Find the coefficients (EWMA, ARCH, GARCH) to maximize the likelihood function

$$f(\mathbf{u};\omega,\alpha,\beta)=\prod_{i=1}^{N}\frac{1}{\sigma_i\sqrt{2\pi}}\exp\left(-\frac{u_i^2}{2\sigma_i^2}\right)$$

- To perform the maximization, likelihood is converted to the log-likelihood (the maximum is the same)

$$LL(\boldsymbol{u};\omega,\alpha,\beta)=\sum_{i=1}^{N}\ln\left[\frac{1}{\sigma_i\sqrt{2\pi}}\exp\left(-\frac{u_i^2}{2\sigma_i^2}\right)\right]=\sum_{i=1}^{N}\ln\left(\frac{1}{\sigma_i\sqrt{2\pi}}\right)-\frac{u_i^2}{2\sigma_i^2}$$

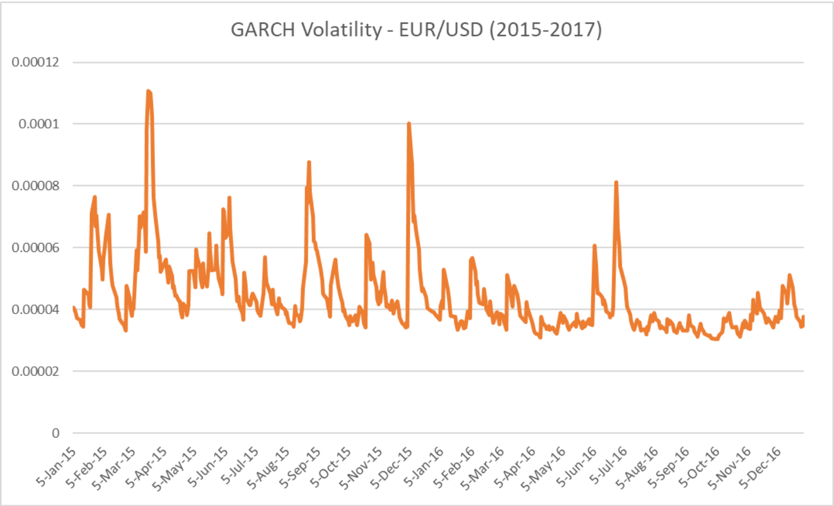
- GARCH usually provides the best results (EWMA does not incorporate mean reversion)

# GARCH Estimation - Example

| | A | B | C | D | E | F | G | H | I |
|---|---|---|---|---|---|---|---|---|---|
| 1 | **Date** | **EURUSD** | **returns** | **sqret** | **GARCHVol** | **LL** | | **GARCH** | **par** |
| 2 | 2-Jan-15 | 1.2003 | 0 | 0 | 4.38E-05 | 0 | | omega | 6.0413E-06 |
| 3 | 5-Jan-15 | 1.1934 | -0.00577 | 3.3E-05 | 4.06E-05 | 3.727562 | | alfa | 0.07323705 |
| 4 | 6-Jan-15 | 1.1889 | -0.00378 | 1.4E-05 | 4.05E-05 | 3.961779 | | beta | 0.78893467 |
| 5 | 7-Jan-15 | 1.1839 | -0.00421 | 1.8E-05 | 3.91E-05 | 3.928928 | | LL | 1788.42601 |
| 6 | 8-Jan-15 | 1.1793 | -0.00389 | 1.5E-05 | 3.82E-05 | 3.969379 | | | |

*Source: Author*

1. Fill the returns time-series and random initial GARCH parameters
2. Compute GARCH predictions of variance for each day
3. Calculate the daily contributions to the log-likelihood and their sum
4. Use Solver to find the values of parameters $\omega$, $\alpha$ and $\beta$ that maximize the total log-likelihood




$$LL(\boldsymbol{u};\omega,\alpha,\beta)=\sum_{i=1}^{N}\ln\left[\frac{1}{\sigma_i\sqrt{2\pi}}\exp\left(-\frac{u_i^2}{2\sigma_i^2}\right)\right]$$

$$\sigma_i^2=\gamma V_L+\alpha u_{i-1}^2+\beta\sigma_{i-1}^2$$

$$\gamma+\beta+\alpha=1$$

$$\omega=\gamma V_L$$

# Forecasting Future Volatility

- Given the GARCH(1,1) parameters at the end of day n-1, estimate

$$\sigma_n^2 = (1-\alpha-\beta)V_L + \alpha u_{n-1}^2 + \beta\sigma_{n-1}^2, \text{ i.e.}$$

$$\sigma_n^2 - V_L = \alpha\left(u_{n-1}^2 - V_L\right) + \beta\left(\sigma_{n-1}^2 - V_L\right)$$

- On day n+k

$$\sigma_{n+k}^2 - V_L = \alpha\left(u_{n+k-1}^2 - V_L\right) + \beta\left(\sigma_{n+k-1}^2 - V_L\right), \text{ and so}$$

$$E[\sigma_{n+k}^2 - V_L] = (\alpha+\beta)E\left[\sigma_{n+k-1}^2 - V_L\right],$$

$$E[\sigma_{n+k}^2] = V_L + (\alpha+\beta)^k\left(\sigma_n^2 - V_L\right)$$

# Volatility Term Structure implied by the model

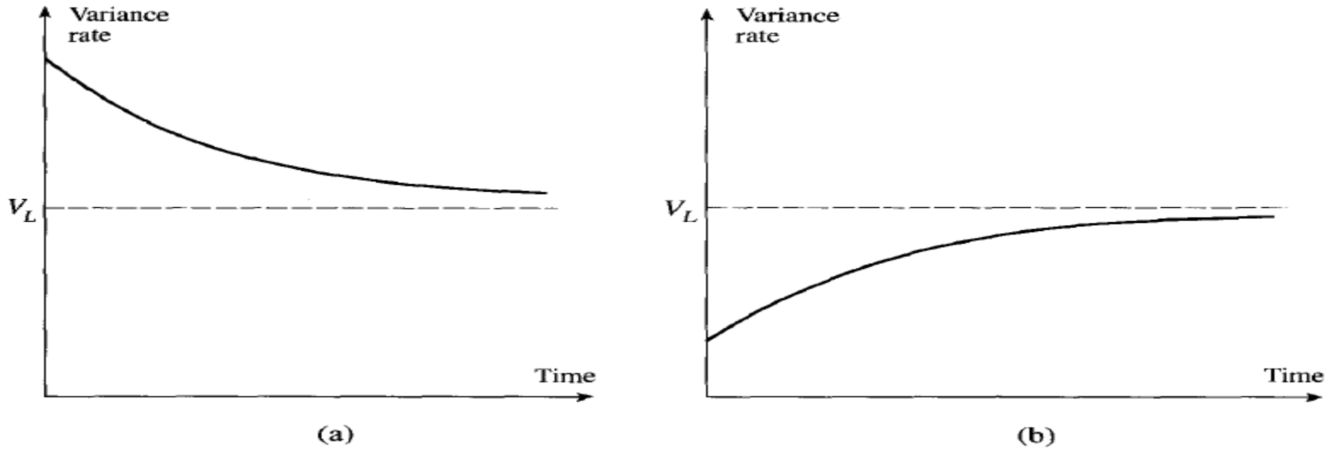



**Figure 17.2** Expected path for the variance rate when (a) current variance rate is above long-term variance rate and (b) current variance rate is below long-term variance rate

Expected (daily) variance over the life of an option $$\frac{1}{N}\sum_{k=0}^{N-1} E(\sigma_{n+k}^2)$$

**Table 17.4** Impact of 1% change in the instantaneous volatility predicted from GARCH(1, 1)

| *Option life (days)* | *10* | *30* | *50* | *100* | *500* |
|---|---|---|---|---|---|
| Option volatility now (%) | 12.03 | 11.61 | 11.35 | 11.01 | 10.65 |
| Option volatility after change (%) | 12.89 | 12.25 | 11.83 | 11.29 | 10.71 |
| Increase in volatility (%) | 0.86 | 0.64 | 0.48 | 0.28 | 0.06 |

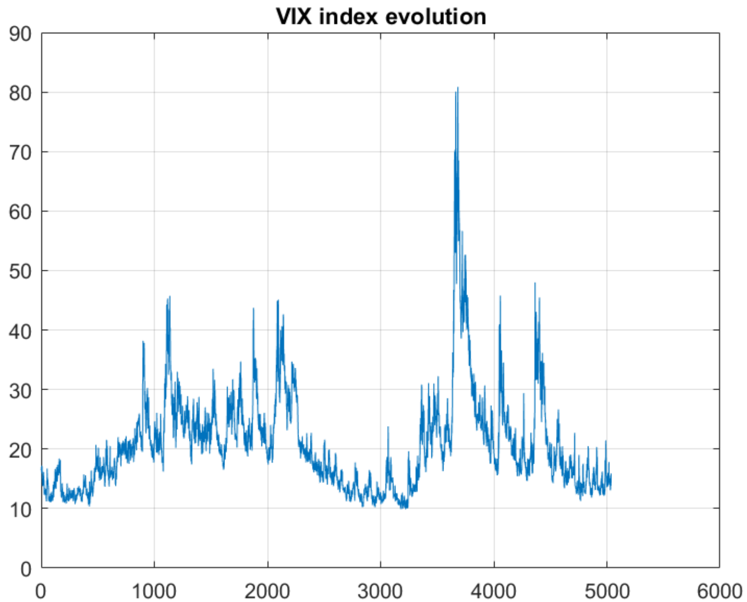
Source: John Hull, Options, Futures, and Other Derivatives, 5th edition

# GARCH extensions

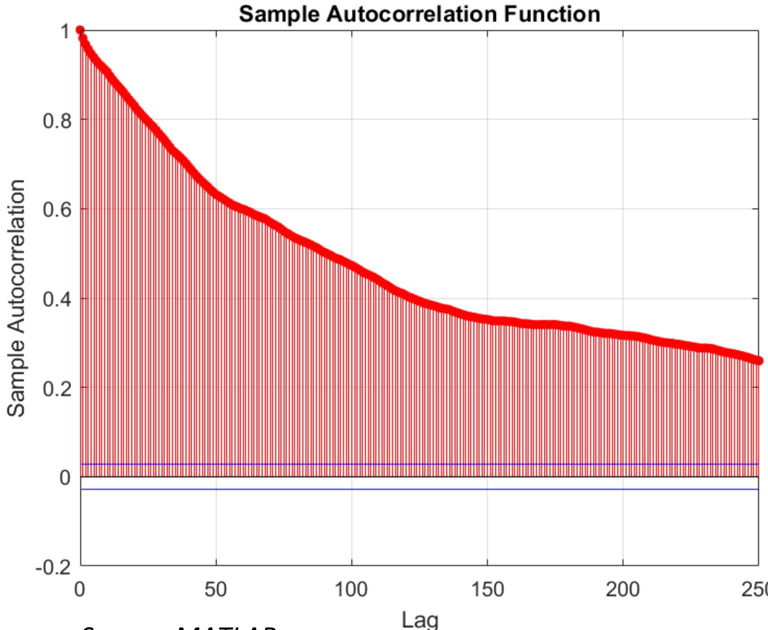
- **FIGARCH** – In practice volatility returns to its mean slower than exponentially, FIGARCH (fractionally integrated GARCH) solves this – hyperbolic decay (i.e. long memory)
- **AGARCH, EGARCH, GJR-GARCH** – Introduce correlation between the price changes and the volatility
- Negative returns have a larger impact on volatility than positive ones (on the stock markets)
- **GJR-GARCH(1,1):**
- $\sigma_t^2 = K + \delta\sigma_{t-1}^2 + \alpha\epsilon_{t-1}^2 + \phi\epsilon_{t-1}^2 I_{t-1}$
- Where $I_{t-1} = 0$ if $\epsilon_{t-1} \geq 0$, and $I_{t-1} = 1$ if$\epsilon_{t-1} < 0$

- For a review see Bollerslev (2009) “Glossary to ARCH (GARCH)“

# Long memory of volatility




*Source: MATLAB*




*Source: MATLAB*

- The long-memory effect can best be seen from the autocorrelation function of the VIX index
- We can see that the autocorrelation function decays in a in a very slow, hyperbolic way, and not the exponential one assumed by short-memory models (e.g. GARCH)

# Estimating Covariances and Correlations

- Standard sample estimate

$$\mathrm{cov}_{jk} = \frac{1}{N}\sum_{i=1}^{N} r_j^i r_k^i \qquad \mathbf{Cov} = \frac{1}{N}\sum_{i=1}^{N} (\mathbf{r}^i)'\mathbf{r}^i$$

- EWMA $\mathbf{Cov} = \frac{1}{c}\sum_{i=1}^{N} \lambda^{N-i}(\mathbf{r}^i)'\mathbf{r}^i, \text{ where } c = \sum_{i=1}^{N}\lambda^{N-i} = \frac{1-\lambda^N}{1-\lambda}$
- Scalar BEKK

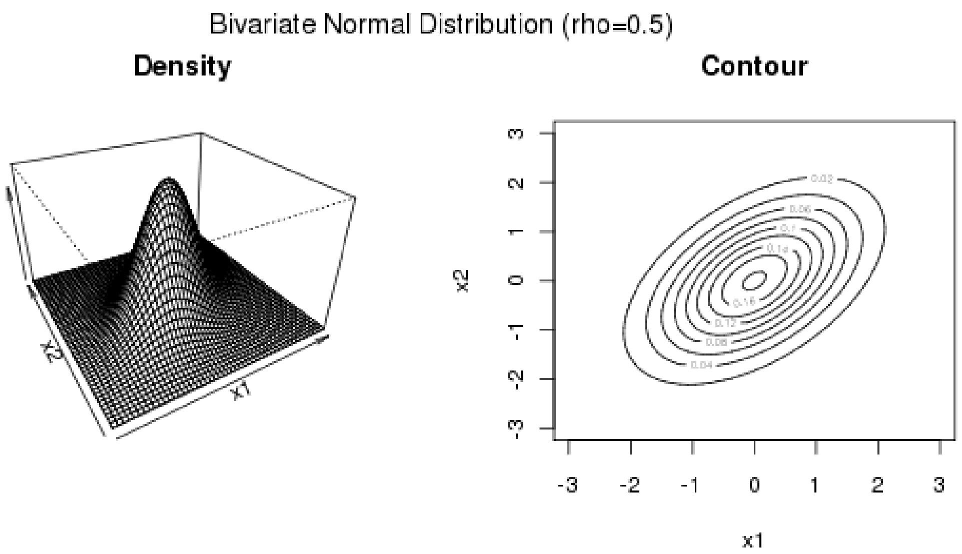
$$\mathbf{Cov}_i = \Omega + \alpha(\mathbf{r}^{i-1})'\mathbf{r}^{i-1} + \beta \mathbf{Cov}_{i-1}$$

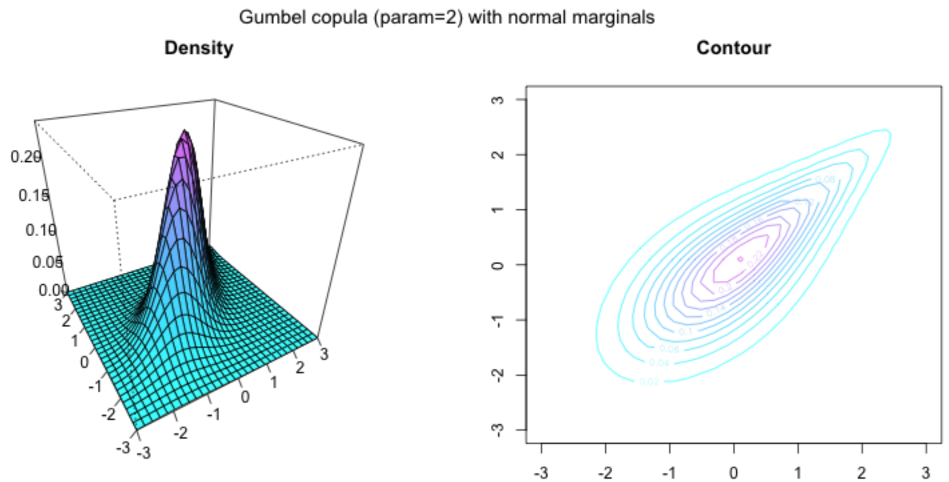
- GARCH BEKK $\mathbf{Cov}_i = C'C + A(\mathbf{r}^{i-1})'\mathbf{r}^{i-1}A' + B\mathbf{Cov}_{i-1}B'$
- DCC GARCH… univariate GARCH to estimate volatilities and scalar BEKK on normalized returns to estimate correlations
- Copula correlations (dependence modeling)

# Copula correlations

- Enable the modeling of **tail correlation**
- Correlations increase in extreme market conditions
- **Sklar's theorem** – Any multivariate distribution $H(x_1, \dots, x_n)$ can be expressed in terms of its marginals $F_i(x) = Pr[X_i \leq x]$ and copula $C$**:**
- $H(x_1, \dots, x_n) = C\left(F_1(x_1), \dots, F_n(x_n)\right)$
- **Gaussian copula:**
- $U_1, \dots, U_n$ is multivariate normal for $U_i = N^{-1}\left(F_i(x_i)\right)$
- **Student-t copula:**
- $U_1, \dots, U_n$ has multivariat t-dist. $U_i = T^{-1}\left(F_i(x_i)\right)$
- **Archimedian copulas:**
- $C_A(u_1, \dots, u_n) = \varphi^{-1}\left(\varphi(u_1) + \cdots + \varphi(u_n)\right)$
- For example **Gumbel copula:** $\varphi(u) = \left(-\ln(u)\right)^{\alpha}$

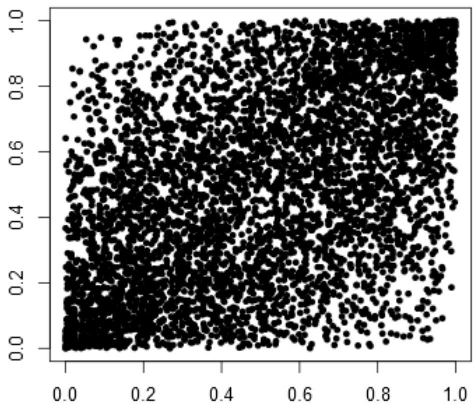
# Illustration – Bivariate Normal Distribution vs. Gumbel copula with Normal marginals



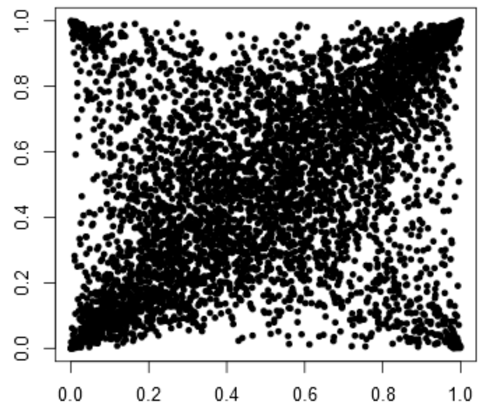




*Source: https://en.wikipedia.org/wiki/Copula_(probability_theory)*

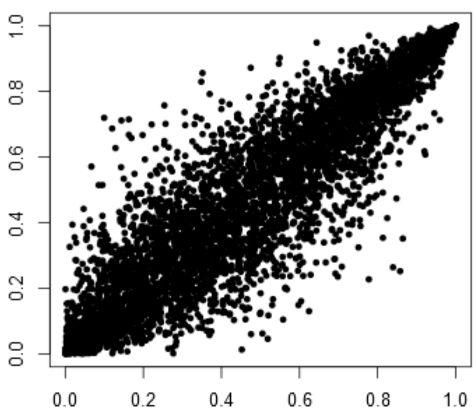
# Illustration – Copulas and tail correlation



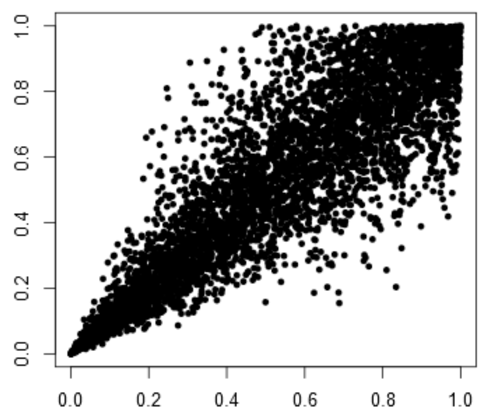
Bivariate Gaussian copula with ρ = 0.5



Bivariate Student-t copula with ρ = 0.5 and dof = 1



Bivariate Gumbel copula with α = 4



Bivariate Clayton copula with α = 5

*Source: https://en.wikipedia.org/wiki/Financial_correlation*

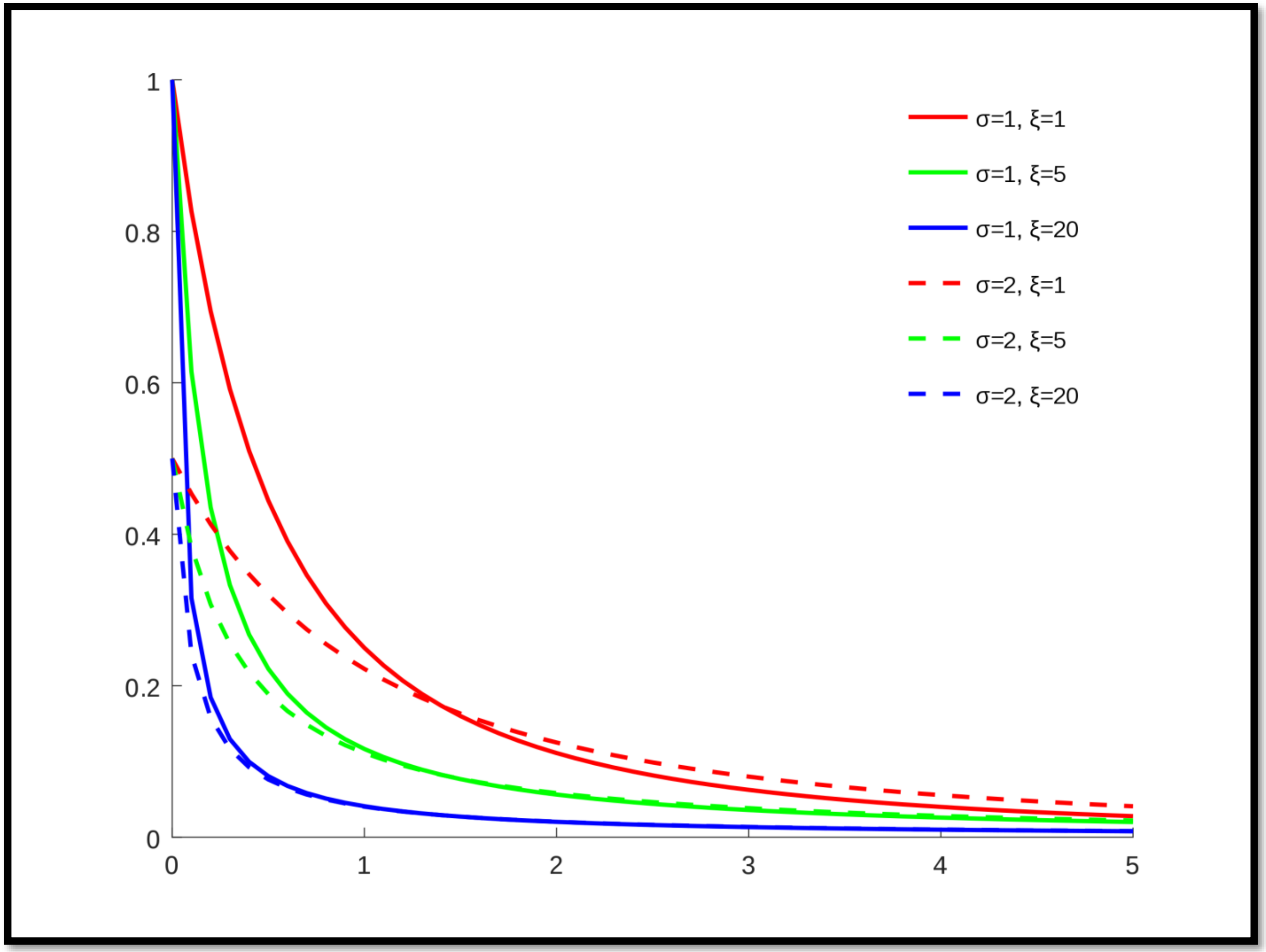
# Copulas for VaR estimation

- Enable the separate modelling of univariate distributions of the asset price time series vs. their interdependencies
- How to proceed:
  1. Model each of the return time-series with an univariate model that best describes its characteristics (i.e. with stochastic volatility and appropriate error term distribution)
  2. Use the univariate model and the cumulative distribution function of the error term distribution to transform each of the time series into a series of uniformly distributed random variables
  3. Apply an appropriate copula model (potentially with dynamic parameters) to capture the inter-dependency between the time series
  4. To compute VaR and cVaR, first simulate uniformly distributed random variables from the copula and then transform them into simulated returns by using the univariate time series models and error term distributions

# Extreme Value Theory (EVT)

- Idea – we can **fit certain distribution only to the tail** of the empirical distribution
- Let $X$ be a los variable (negative return) with distribution $F$, and define **excess losses** as:
- $F_u(y) = P[X - u \leq y | X > u] = \frac{F(u+y) - F(u)}{1 - F(u)}$
- According to Gnedenko, for a wide class of $F$ the excess function $F_u(y)$ converges to the **generalized Pareto cumulative distribution (GPD)**
- We can then use the GPD to estimate:
- $\Pr[X > x] \cong (1 - F(u))\left(1 - G_{\xi,\beta(u)}(x - u)\right)$
- We can then solve $\Pr[X > VaR] = 1 - \alpha$
- GPD also enables to calculate the expected shortfall

# Generalized Pareto Distribution




*Source: https://en.wikipedia.org/wiki/Generalized_Pareto_distribution*

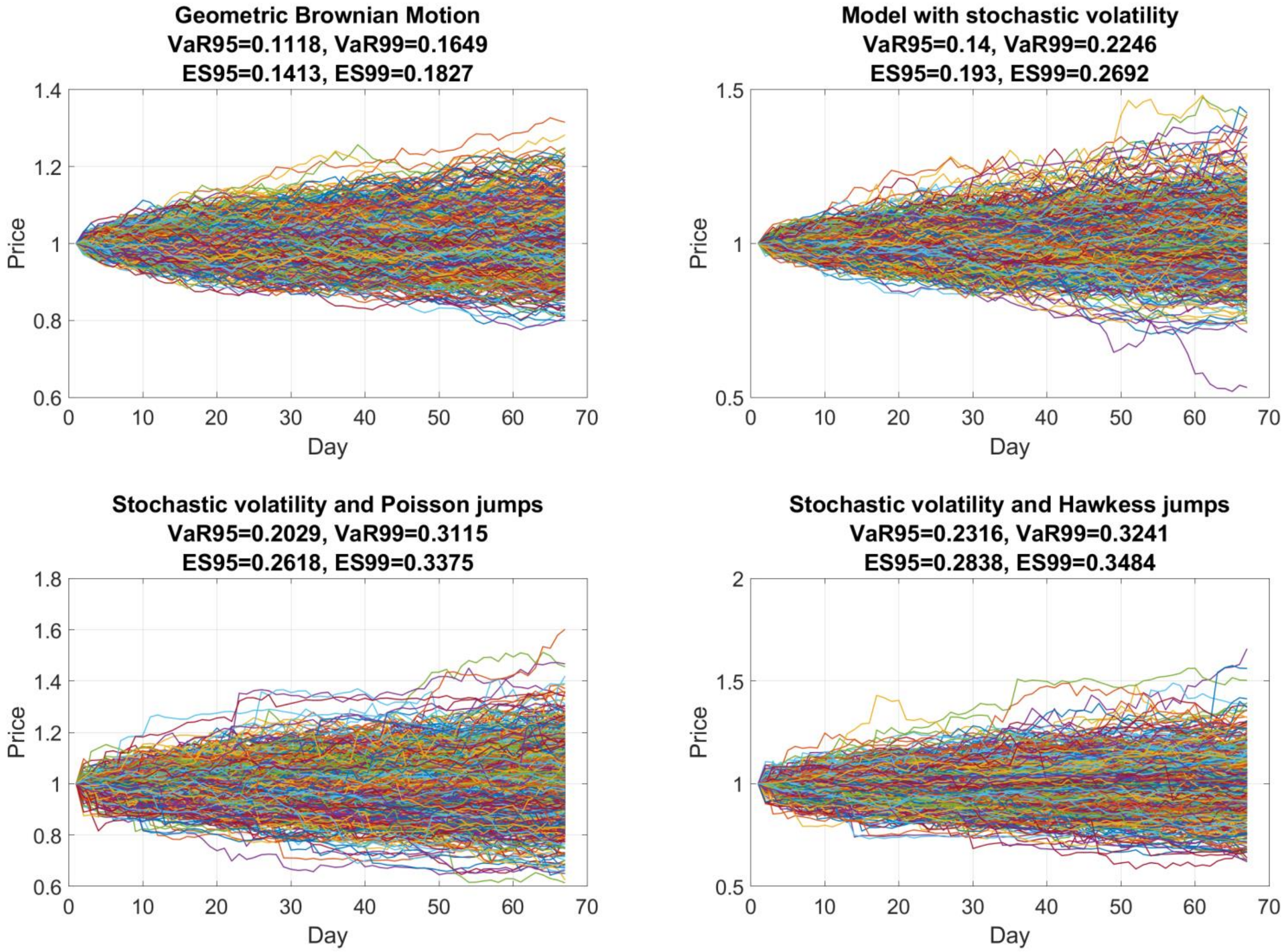
# Stochastic-Volatility Jump-Diffusion

- Simulation of the market factors with SVJD processes
- **Stochastic volatility** – Increases the tails of the return distribution in longer horizons
- **Jumps** – Increase the tails of the return distribution in shorter horizons
- Example – **Log-Variance model with Poisson jumps**
- Log-Price process: $dp(t) = \mu dt + \sigma(t)dz(t) + j(t)dq(t)$
- Log-Variance process: $dh(t) = \kappa[\theta - h(t)]dt + \xi dz_V(t)$
- Where: $h(t) = \ln[\sigma^2(t)]$ $j(t) \sim N(\mu_J, \sigma_J)$ $\mathrm{Pr}[\mathrm{dq(t)} = 1] = \lambda \mathrm{dt}$
- The model can further assume correlation between $dz$ and $dz_V$, time-variability of λ, or jumps in $h(t)$
- Parameter estimation methods like MCMC

# SVJD models - Example

- Four types of stochastic processes were fitted to the EUR-USD historical returns time series:
  1. Geometric Brownian Motion
  2. Log-SV model
  3. SVJD model with Poisson jumps
  4. SVJD model with Hawkess jumps
- Value at Risk and Expected Shortfall were then computed by simulating the processes over the 3-month period into the future

# 3-Month VaR estimation with SVJD models




*Source: MATLAB*

EVROPSKÁ UNIE
Evropské strukturální a investiční fondy
Operační program Výzkum, vývoj a vzdělávání

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