Financial Derivatives II
Part 2

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Content

- ✓ Introduction – overview of B.-S. option pricing and hedging
- ✓ Market Risk Management
- ✓ Estimating volatilities and correlations
- ➢ Interest Rate Derivatives Pricing- Martingale and measures
- Standard Market Model

Content

- Convexity, time, and quanto adjustments
- Short-rate and advanced interest rate models
- Volatility smiles
- Exotic options
- Alternative stochastic models
- Numerical methods for option pricing
- Credit derivatives

Martingale and measures (interest rate derivatives pricing, NSA approach)

Martingales, Measures, and Numeraires

- Interest rates cannot be constant (or deterministic) valuing interest rate derivatives!!!
- Can we still evaluate derivatives taking the expected payoff and discounting it at the risk-free rate???
- Yes, but a different “risk-neutral measure” must be used!!!
- For example, we would like to make $\frac{f_t}{P(t,T)}$ to be a martingale

General risk-neutral probabilities

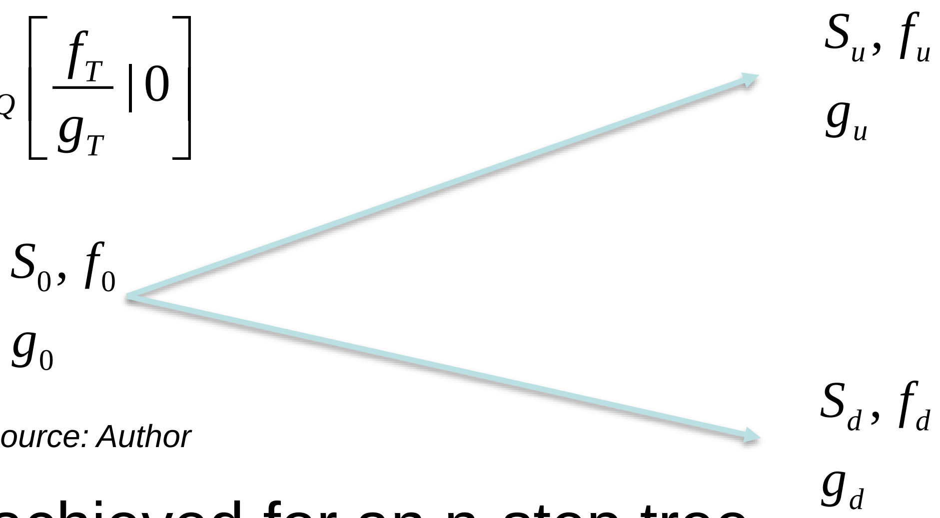
- A general discount factor g … **numeraire**
- Define the risk neutral probability so that

$$Z_0 = qZ_u + (1-q)Z_d \text{ where } Z = \frac{S}{g}$$

- And use the replication argument to show that f/g is a **martingale**

$$\frac{f_0}{g_0} = q\frac{f_u}{g_u} + (1-q)\frac{f_d}{g_d} = \mathrm{E}_Q\left[\frac{f_T}{g_T} \mid 0\right]$$

as $f = \alpha S + \beta g$



Source: Author

- The same can be achieved for an n-step tree

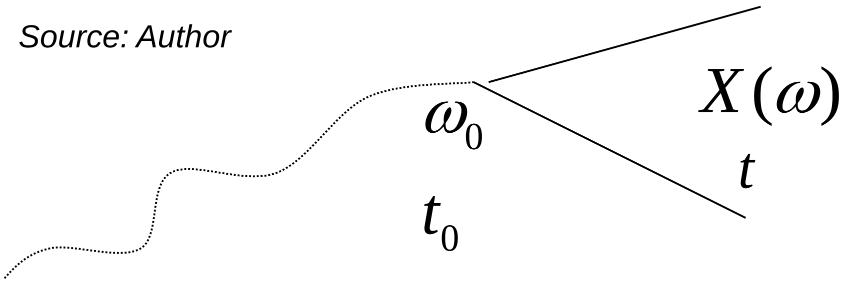
Binomial Trees with Infinitesimals

- It has been shown (Cox, Ross, Rubinstein) that the values obtained using n-step binomial trees converge to the B.-S. value
- Binomial trees are in practice used for numerical approximations of values of American and exotic options
- Continuous trading in fact does not exist, real trading is always discrete. Are not discrete models with small steps better approximations of the reality than continuous models???
- (Cutland, Kopp, Willinger) Binomial Trees with infinitesimals provide (up to an infinitesimal error) the B-S value

Important Notions Defined on Binomial Trees

- Conditional expectation

Source: Author



$$E[X \mid \omega_0] = \frac{1}{P(\omega_0)} \sum_{\omega, t_0 = \omega_0} X(\omega) P(\omega)$$

- **Martingale**: $X(\omega_0) = E[X \mid \omega_0]$ for every ω_0
- Markov process $E[f(X) \mid \omega_0]$ depends only on $X(\omega_0)$
- stochastic integral, SDE, replication by a strategy, risk-neutral measure

See e.g. S.Shreve: The Binomial Asset Pricing Model

Market Price of Risk

- **Proposition:** All derivatives following the price process of the form $dg = \mu g dt + \sigma g dz$ have the same price of risk defined as
$$\lambda = \frac{\mu - r}{\sigma}$$
where r is the risk-free rate.
- *Proof* uses a similar arbitrage argument as in the B-S model. Given two derivative securities with the same source of risk combine them to eliminate the risk in a short time interval dt. The fact that the portfolio yields the risk-free return leads to the equation between the corresponding prices of risk.
- Can be generalized for n sources of uncertainty

Market Price of Risk equality - Proof

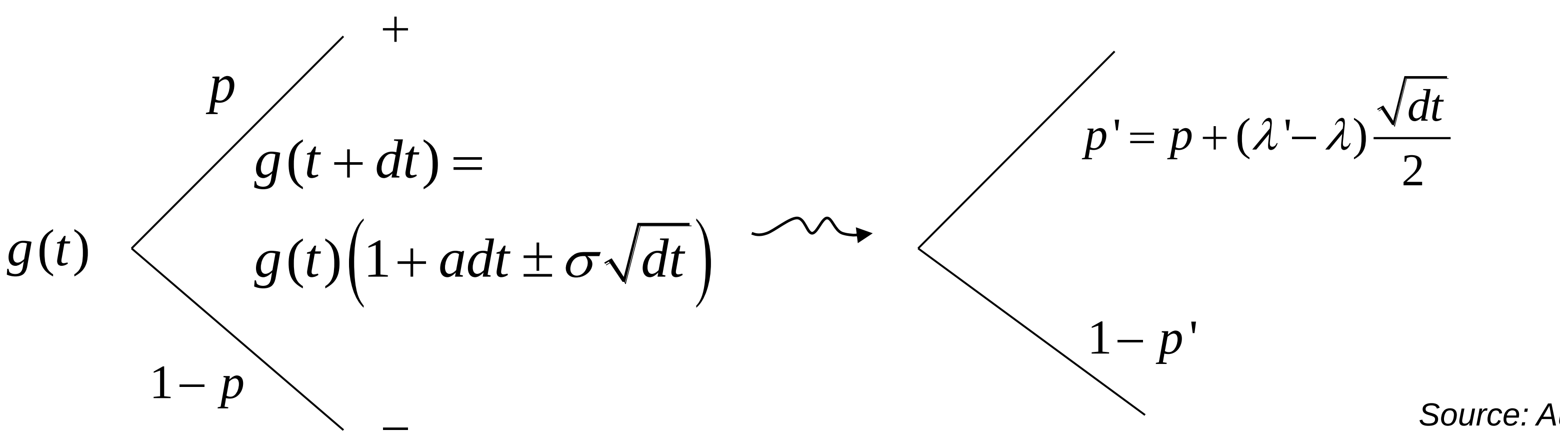
- We have 2 derivatives with the same source of risk:
- $df_1 = \mu_1 f_1 dt + \sigma_1 f_1 dz$
- $df_2 = \mu_2 f_2 dt + \sigma_2 f_2 dz$
- We can construct a risk-less portfolio by entering into $\sigma_2 f_2$ units of f_1 and $-\sigma_1 f_1$ units of f_2
- $\Pi = (\sigma_2 f_2) f_1 - (\sigma_1 f_1) f_2$
- The portfolio value will then change according to:
- $d\Pi = (\sigma_2 f_2) df_1 - (\sigma_1 f_1) df_2$
- $d\Pi = (\sigma_2 f_2)(\mu_1 f_1 dt + \sigma_1 f_1 dz) - (\sigma_1 f_1)(\mu_2 f_2 dt + \sigma_2 f_2 dz)$
- $d\Pi = (\sigma_2 \mu_1 - \sigma_1 \mu_2) f_1 f_2 dt$
- Since Π is risk-less it must earn the risk-free return
- $d\Pi = r\Pi dt = r(\sigma_2 - \sigma_1) f_1 f_2 dt$
- So we get the following equality
- $(\sigma_2 \mu_1 - \sigma_1 \mu_2) f_1 f_2 dt = r(\sigma_2 - \sigma_1) f_1 f_2 dt$
- $\sigma_2 \mu_1 - \sigma_1 \mu_2 = r(\sigma_2 - \sigma_1)$
- $\frac{\mu_1 - r}{\sigma_1} = \frac{\mu_2 - r}{\sigma_2}$
- The price of risk is same for all derivatives with same sources of risk

We define *price of risk* (Sharpe ratio) as:

$$\lambda = \frac{\mu - r}{\sigma}$$

Equivalent Martingale Measure Using Infinitesimals

- Price of Risk: Assume that g has only one source of uncertainty $dg = \mu g dt + \sigma g dz$
- Define the price of risk as $\lambda = \dfrac{\mu - r}{\sigma}$
- Let λ’>0 be any other price of risk, then we can change the measure accordingly



Source: Author

Change of probability

$$E'\left[\frac{dg}{g}\right]=\left(p+\gamma\frac{\sqrt{dt}}{2}\right)\left(adt+\sigma\sqrt{dt}\right)+\left(1-p-\gamma\frac{\sqrt{dt}}{2}\right)\left(adt-\sigma\sqrt{dt}\right)=$$

$$=E\left[\frac{dg}{g}\right]+2\gamma\frac{\sqrt{dt}}{2}\sigma\sqrt{dt}=E\left[\frac{dg}{g}\right]+\gamma\sigma dt$$

$$=\left(r+\lambda\sigma+(\lambda'-\lambda)\sigma\right)dt=\left(r+\lambda'\sigma\right)dt$$

$$\gamma=\lambda'-\lambda$$

$$a=\mu=r+\lambda\sigma$$

$$E'\left[\left(\frac{dg}{g}\right)^2\right]=\left(p+\gamma\frac{\sqrt{dt}}{2}\right)\left(adt+\sigma\sqrt{dt}\right)^2+\left(1-p-\gamma\frac{\sqrt{dt}}{2}\right)\left(adt-\sigma\sqrt{dt}\right)^2=$$

$$=E\left[\left(\frac{dg}{g}\right)^2\right]+\gamma\frac{\sqrt{dt}}{2}4adt\sigma\sqrt{dt}=E\left[\left(\frac{dg}{g}\right)^2\right]+2\gamma a\sigma dt^2$$

$$\text{var}'\left[\frac{dg}{g}\right]=E'\left[\left(\frac{dg}{g}\right)^2\right]-E'\left[\frac{dg}{g}\right]^2=\text{var}\left[\frac{dg}{g}\right]+o(dt)=\sigma^2dt+o(dt)$$

Source: Author

Change of Price of Risk

- The previous results show that we can change the drift and the price of risk, while not changing the variance
- Assume a stochastic proces: $dg = \mu g dt + \sigma g dz$
- In order to change the price of risk $\lambda = (\mu - r)/\sigma$ to an arbitrary λ', we need to change $\mu = r + \lambda\sigma$ to
$$\mu' = r + \lambda'\sigma = \mu + (\lambda' - \lambda)\sigma$$
- Probability p in the binomial tree wil change to q:
- $q = 0.5 + \frac{\mu'}{2\sigma}\sqrt{dt} = 0.5 + \frac{\mu + (\lambda' - \lambda)\sigma}{2\sigma}\sqrt{dt} = p + \frac{\lambda' - \lambda}{2}\sqrt{dt}$

Change of Numeraire – Equivalent Martingale Measure

- **Numeraire** is a security (stochastic process) attaining positive values used as a unit to measure values of other securities.
- **Theorem:** If g is a numeraire than there is a measure (*equivalent martingale measure* determined by a price of risk) so that for any security (stochastic process) f with the same sources of uncertainty f/g is a martingale.
- *Proof:* Use the Ito lemma applied to $\ln(f)$, $\ln(g)$, and $\ln(f/g)=\ln(f)-\ln(g)$ to show that if σ_g is the new price of risk then f/g has zero drift, i.e. is a martingale.

Equivalent Martingale Measure Using Infinitesimals

- Recall that we have shown that all securities with the same sources of uncertanity must have the same price of risk in an equilibrium (non arbitrage) market
- Show that $\lambda'=\sigma_g$ gives the equivalent martingale measure with respect to g
- This is done, e.g., using the **Ito's lemma** which is easily proved using infinitesimals as $dz^2= dt$

$$dx = a(x,t)dt + b(x,t)dz, \quad G = G(x,t)$$

$$dG = \frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial t}dt + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}dx^2 + \cdots = \left(\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2\right)dt + \frac{\partial G}{\partial x}bdz$$

Equivalent Martingale Measure Using Infinitisemals - Proof

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$$\lambda = \frac{\mu_g - r}{\sigma_g}$$

$$\mu' = r + \lambda' \sigma$$

- Assume that the numeraire $g(\mathrm{t})$ follows a process:
- $dg = \mu_g g dt + \sigma_g g dz$ (under the measure *P*)
- Let λ be the price of risk, so that $\mu_g = r + \lambda \sigma_g$
- Changing λ to $\lambda' = \sigma_g$ will change the drift rate to $r + \sigma_g^2$
- $dg = \left(r + \sigma_g^2\right) g dt + \sigma_g g dz$ (under the measure *Q*)
- Let f be a derivative following a proces (under the measure *Q*):
- $df = \left(r + \sigma_g \sigma_f\right) f dt + \sigma_f f dz$
- To prove that f/g is martingale, we apply the Ito's Lemma to get:
- $d(\ln(g)) = \left(r + \sigma_g^2 - \sigma_g^2/2\right) dt + \sigma_g dz = \left(r + \sigma_g^2/2\right) dt + \sigma_g dz$
- $d(\ln(f)) = \left(r + \sigma_g \sigma_f - \sigma_f^2/2\right) dt + \sigma_f dz$
- By subtracting the two equations we get:
- $d(\ln(f/g)) = d\left(\ln(f) - \ln(g)\right) = \left(\sigma_g \sigma_f - \sigma_f^2/2 - \sigma_g^2/2\right) dt + \left(\sigma_f - \sigma_g\right) dz$
- $d(\ln(f/g)) = d\left(\ln(f) - \ln(g)\right) = -\frac{1}{2}\left(\sigma_f - \sigma_g\right)^2 dt + \left(\sigma_f - \sigma_g\right) dz$
- We need to apply Ito's Lemma again to $f/g = \exp(\ln(f/g))$
- $d(f/g) = \left(\sigma_f - \sigma_g\right)(f/g) dz$ <u>which is a martingale</u>

Applications of Equivalent Martingale Measures

- In particular

$$\frac{f(0)}{g(0)} = E_g\left[\frac{f(T)}{g(T)}\right]$$

$$f(0) = g(0)E_g\left[\frac{f(T)}{g(T)}\right]$$

MM account:

$$g(T) = \exp\left(\int_0^T r(s)ds\right)$$

- Examples of numeraires: money market account, zero coupon bond, annuity

Zero-bond:

$$P(0,T) = e^{-R_T T}$$

$$f(0) = P(0,T)E_T\left[\frac{f(T)}{P(T,T)}\right] = P(0,T)E_T[f(T)]$$

- Hence the value of *f* can be calculated as a „discounted“ expected value of the payoff

Source: Author

Standard Market Model

- Assumes that the underlying variable is lognormally distributed under the r.n. measure
- In particular if, w.r.t. the $P(t,T)$ risk-neutral measure, $\ln S_T \sim N\left(\ln E[S_T] - \sigma^2 T/2, \sigma^2 T\right)$
- then

$$f_0 = P(0,T) E_T\left[\max(S_T - K, 0)\right]$$

$$E_T\left[\max(S_T - K, 0)\right] = E_T[S_T] N(d_1) - K N(d_2)$$

$$d_1 = \frac{\ln\left(E_T[S_T]/K\right) + \sigma^2 T/2}{\sigma\sqrt{T}} \qquad d_2 = \frac{\ln\left(E_T[S_T]/K\right) - \sigma^2 T/2}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

Recall Derivation of the BS Formula (for a European Call Option)

Our goal is to calculate $E[\max(S-K,0)] = \int_K^\infty (S-K)g(S)dS$ with $S = S_T$

$$\ln S \sim N\left(m, w^2\right), \text{ where } m = \ln S_0 + \left(r - \frac{1}{2}\sigma^2\right)T \text{ and } w^2 = \sigma^2 T.$$

Substitute $X = \dfrac{\ln S - m}{w}$ and so $g(S)dS = \varphi(X)dX = \dfrac{1}{\sqrt{2\pi}} e^{-X^2/2} dX$

$$E[\max(S-K,0)] = \int_{(\ln K - m)/w}^{\infty} (e^{Xw+m} - K)\varphi(X)dX =$$

$$= \int_{(\ln K - m)/w}^{\infty} \frac{1}{\sqrt{2\pi}} e^{(-X^2+2Xw+2m)/2} dX - K \int_{(\ln K - m)/w}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-X^2/2} dX$$

The second integral is easy:

$$\int_{(\ln K - m)/w}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-X^2/2} dX = N(-(\ln K - m)/w) \qquad N(x) = \Phi(x) = \Pr[X \le x] = \int_{-\infty}^{x} \varphi(X)dX$$

Derivation of the BS Formula

Regarding the first integral:

$$\frac{-X^2+2Xw+2m}{2}=\frac{-(X-w)^2+2m+w^2}{2}$$

$$\int_{(\ln K-m)/w}^{\infty}\frac{1}{\sqrt{2\pi}}e^{(-X^2+2Xw+2m)/2}dX=e^{m+w^2/2}\int_{(\ln K-m)/w}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-(X-w)^2/2}dX=$$

$$=e^{m+w^2/2}N(w-(\ln K-m)/w).$$

It is easy to check:

$$w-(\ln K-m)/w=\frac{-(\ln K-m)+w^2}{w}=\frac{-\ln K+\ln S_0+rT-\sigma^2T/2+\sigma^2T}{\sigma\sqrt{T}}=$$

$$=\frac{\ln S_0/K+(r+\sigma^2T)/2}{\sigma\sqrt{T}}=d_1,$$

$$-(\ln K-m)/w=\frac{\ln S_0/K+(r-\sigma^2T)/2}{\sigma\sqrt{T}}=d_2,\qquad e^{m+w^2/2}=e^{\ln S_0+rT}=S_0e^{rT}$$

And so $\boxed{c=e^{-rT}\left(S_0e^{rT}N(d_1)-KN(d_2)\right)=S_0N(d_1)-e^{-rT}KN(d_2)}$

Source: Author

B.-S. Formula with Stochastic Interest Rates

- European call on a non dividend stock with maturity T, then

$$E_T[S_T] = E_T\left[\frac{S_T}{P(T,T)}\right] = \frac{S_0}{P(0,T)}$$

- Where the expectation is taken with the measure risk-neutral w.r.t. $P(t,T)$
- If (!!!) we assume that ln S_T is normal („*standard market model*“) and R is the maturity T interest rate then

$$c_0 = S_0 N(d_1) - e^{-RT} K N(d_2)$$

$$d_1 = \frac{\ln\left(S_0/K\right) + \left(R + \sigma^2/2\right)T}{\sigma\sqrt{T}}$$

$$P(0,T) = e^{-RT}$$

$$d_2 = \frac{\ln\left(S_0/K\right) + \left(R - \sigma^2/2\right)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

Black's formula

- Uses <u>futures/forward prices as the key input</u>
- For options on income paying assets and commodities (or for options on futures) it is more appropriate to use the Blacks formula based on

$$0 = P(0,T)E[S_T - F_0] = P(0,T)\left(E[S_T] - F_0\right)$$

$$E_T[S_T] = F_0$$

- Since $S_T = F_T$ the formula can be stated in terms of forward/futures price volatility

$$c_0 = P(0,T)\left(F_0 N(d_1) - KN(d_2)\right)$$

$$d_1 = \frac{\ln\left(F_0 / K\right) + \sigma_F^2 T / 2}{\sigma_F \sqrt{T}} \qquad d_2 = \frac{\ln\left(F_0 / K\right) - \sigma_F^2 T / 2}{\sigma_F \sqrt{T}} = d_1 - \sigma_F \sqrt{T}$$

Option to Exchange One Asset for Another

- Option to exchange one asset *U* for another asset *V* at time T (e.g. convertible bonds)

$$f_T = \max(V_T - U_T, 0)$$

- Let the numeraire = U, then

$$f_0 = U_0 E_U\left[\frac{\max\left(V_T - U_T, 0\right)}{U_T}\right] = U_0 E_U\left[\max\left(\frac{V_T}{U_T} - 1, 0\right)\right]$$

$$E_U\left[\frac{V_T}{U_T}\right] = \frac{V_0}{U_0}$$

- Assuming lognormality of U and V

$$f_0 = V_0 N(d_1) - U_0 N(d_2)$$

$$d_1 = \frac{\ln\left(V_0 / U_0\right) + \sigma_h^2 T / 2}{\sigma_h \sqrt{T}}$$

$$\sigma_h = \sqrt{\sigma_V^2 - 2\rho\sigma_V\sigma_U + \sigma_U^2}$$

$$d_2 = \frac{\ln\left(V_0 / U_0\right) - \sigma_h^2 T / 2}{\sigma_h \sqrt{T}} = d_1 - \sigma_h\sqrt{T}$$

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Standard Market (Black's) Model for interest rate options

- Applicable to bond options, interest rate caps/floors, and to swap options (swaptions)
- Generally use the $P(t,T)$ forward neutral measure and the assumption of lognormality of the underlying variable V_T
- If $E_T[V_T] = F_0$ and if the standard deviation of $\ln V_T$ is $\sigma\sqrt{T}$ then we get the "standard formulas", e.g. for a European call option:

$$c_0 = P(0,T)\left(F_0 N(d_1) - KN(d_2)\right)$$

$$d_1 = \frac{\ln\left(F_0 / K\right) + \sigma_F^2 T / 2}{\sigma_F \sqrt{T}}$$

$$d_2 = \frac{\ln\left(F_0 / K\right) - \sigma_F^2 T / 2}{\sigma_F \sqrt{T}} = d_1 - \sigma_F \sqrt{T}$$

Bond Options

- OTC bond options, embedded options in callable/puttable bonds, loan prepayment options and loan commitments
- The underlying variable = Q_T the cash bond price, the bond forward value

$$F_0 = \frac{Q_0 - I}{P(0,T)}$$

- where I is the present value of coupons to be paid (not AI)
- Alternatively underlying could be the net price

Bond Volatility

- The standard deviation of ln $Q_T = \sigma_Q\sqrt{T}$ depends on $\sqrt{T}$ and on the bond duration
- Note that we are estimating time T bond price volatility
- σ_Q can be estimated from the yield volatility using the concept of duration

$$\frac{\Delta Q}{Q_0} \cong -D_0 \Delta y = -D_0 y_0 \frac{\Delta y}{y_0}$$

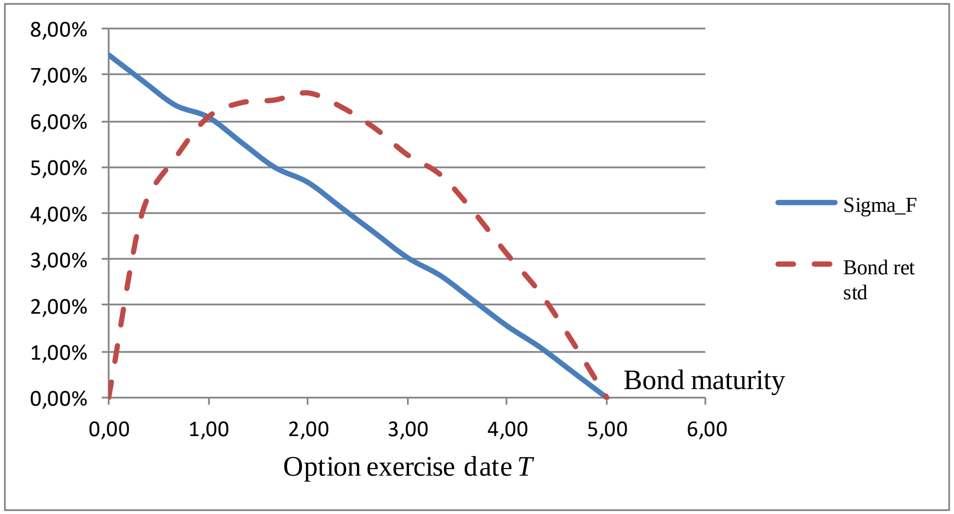
$$\sigma_Q \cong D_0 y_0 \sigma_y$$

Time T volatility

$$\sigma_F \cong D_F y_F \sigma_y$$

$$\text{std}(\ln Q_T) \cong D_F y_F \sigma_y \sqrt{T}$$

$$\cong y_F \sigma_y (T_M - T)\sqrt{T}$$

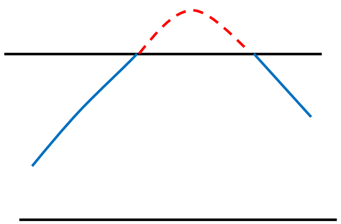


Source: Author

Caps and Floors

- Interest rate cap payoff can be expressed as a set of payoffs of individual caplets

$$\max(R_{M,i} - K_u, 0) \text{ paid at } t_{i+1}$$



Source: Author

- Similarly a floor can be decomposed into floorlets.
- Collar is defined as a long position in a cap and a short position in a floor with the same underlying and payment times (strike floor < strike cap)
- Note that Value of cap = Value of floor + Value of swap …put-call parity…with the same strike

$$cap(K) - floor(K) = irs(K)$$

Valuation of caps and floors

- The caplets and floorlets can be valued independently
- The rate observed at t_i is payable at t_{i+1}, hence we need to use $P(t, t_{i+1})$ forward risk neutral measure, so

$$F_i = E_{t_{i+1}}\left[R_{M,i}\right]$$

and

$$c_i = L\delta_i P(0,t_{i+1})\left(F_i N(d_1) - K_u N(d_2)\right)$$

$$d_1 = \frac{\ln\left(F_i / K_u\right) + \sigma_i^2 t_i / 2}{\sigma_i \sqrt{t_i}}$$

$$d_2 = \frac{\ln\left(F_i / K_u\right) - \sigma_i^2 t_i / 2}{\sigma_i \sqrt{t_i}} = d_1 - \sigma_i \sqrt{t_i}$$

Cap/Floor Volatilities

- Each caplet/floorlet could be valued with individual (*spot*) volatility corresponding to the option maturity
- Alternative possibility (used by the market) is to use a single (flat) volatility for all caplets in a cap

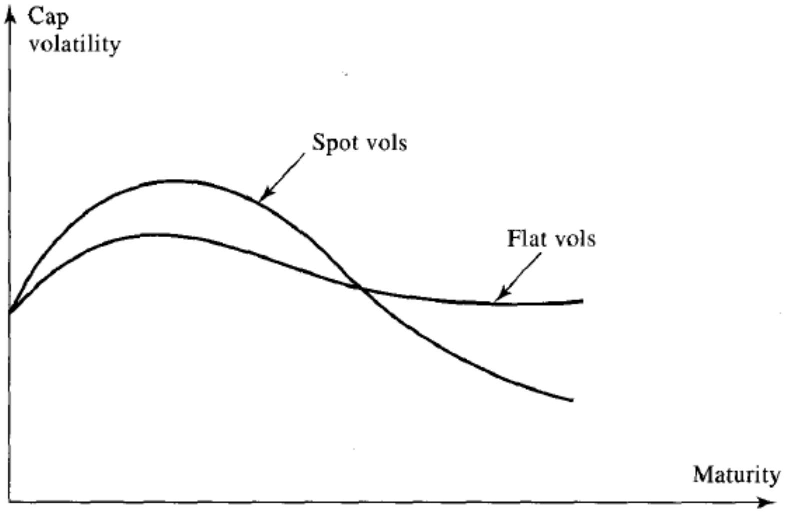


Figure 22.3 The volatility hump

Source: John Hull, Options, Futures, and Other Derivatives, 5th edition

Cap/Floor Quotations

EURCAP=TKFX Totan ICAP-TOK LINKED DISPLAYS MONEY

TOTAN ICAP

	EUR	See<TKFXINFO>			DEALING
1Y	56.3	57.3	Totan ICAP	TOK	15:33
2Y	57.9	58.9	Totan ICAP	TOK	15:33
3Y	47.1	48.1	Totan ICAP	TOK	15:33
4Y	46.6	47.6	Totan ICAP	TOK	15:33
5Y	44.2	45.2	Totan ICAP	TOK	15:33
7Y	38.3	39.3	Totan ICAP	TOK	15:33
10Y	32.6	33.6	Totan ICAP	TOK	15:33

CAPS/FLOORS

Updated at 20:13:24

Currency: EUR Trade Date: 12 2 2012

Type: Sell Collar Vanilla

Main | Volatilities | Caplets and Floorlets | Amortization | ZC Curve

Start Date	Cap Strike	Floor Strike	Cap Vol	Floor Vol	Forward	Premium	Notional	Delta	Gamma
14 2 2012	1,80000	0,80000	59,76	55,61	1,365	0,00	1 000 000,00	0,0000	0,0000
14 8 2012	1,80000	0,80000	59,76	55,61	1,004	-200,38	1 000 000,00	0,3415	-27,1703
14 2 2013	1,80000	0,80000	61,66	59,34	0,809	-742,99	1 000 000,00	0,5382	-29,6361
14 8 2013	1,80000	0,80000	62,50	63,21	1,403	974,03	1 000 000,00	0,6560	17,2828

Source: Author

Swaptions

- Options to enter into a certain interest rate swap at a certain time in the future
- Similarly to caps and floors, can be used as an interest rate management instrument sold to corporations
- Could be equivalently viewed as an option on the fixed coupon bond with the strike equal to the nominal

Valuation of European Swaptions

- Use the Black's model with the assumption that s_T is lognormal
- The payoff (fix-payer) $$f_T = \sum_{i=1}^{N} P(T,T_i)\delta_i L \max\left(s_T - s_K, 0\right)$$
- To justify the following we in fact need the annuity risk neutral measure $g(t)=A(t)$!!!

$$c = LA(0)(s_0 N(d_1) - s_K N(d_2))$$

$$d_1 = \frac{\ln\left(s_0 / s_K\right) + \sigma_F^2 T / 2}{\sigma_F \sqrt{T}}$$

$$A(t) = \sum_{i=1}^{N} \delta_i P(t,T_i)$$

$$d_2 = \frac{\ln\left(s_0 / s_K\right) - \sigma_F^2 T / 2}{\sigma_F \sqrt{T}} = d_1 - \sigma_F \sqrt{T}$$

s_0...forward swap rate

Remark: Swaptions can be also valued as bond options, note that the two Black's models are not mutually consistent

Swaption Volatility Quotations

- Two dimensions: exercise data and the swap tenor

SWAPTION VOLATILITY

TTKL	1Y	2Y	3Y	4Y	5Y	7Y	10Y
1M EX	51.20	42.90	45.10	43.10	40.30	36.30	34.70
3M EX	55.10	44.70	46.00	43.80	41.00	36.80	35.00
6M EX	57.50	46.40	46.10	43.20	41.00	37.00	35.40
1Y EX	60.60	47.10	45.30	42.70	40.40	36.70	35.00
2Y EX	57.40	44.40	40.30	37.70	36.20	33.80	32.30
3Y EX	47.50	38.50	35.60	33.70	32.50	31.00	29.80
4Y EX	38.70	33.00	31.40	30.40	29.70	28.60	27.70
5Y EX	33.20	29.70	28.80	28.10	27.60	26.80	26.40
7Y EX	27.70	26.20	25.70	25.20	24.80	24.60	25.20
10Y EX	23.20	22.80	22.80	22.90	23.10	23.80	24.90
15Y EX	24.30	24.60	25.00	25.40	25.90	26.80	28.20
20Y EX	27.40	28.50	29.00	29.50	29.90	30.60	31.10

Source: Author

Swaption Valuation Example

- Value at-the-money swaption with 1Y exercise on a 5Y swap with 1 mil EUR principal, if the forward swap rate is 1.96%.
- The volatility quotes are given on the previous slide and the actual annuity value is 4.7.
- The table has two dimensions, exercise times and tenors, thus the volatility corresponding to our swaption is 40.4%.
- According to the formula and using the given parameters, we obtain 17 447 EUR from the perspective of the fix-rate payer.

Negative Interest Rates and the Standard Market Model

- The lognormal model is not consistent with negative interest rates

EURD= EUR Deposits

RIC		Bid/Ask	Contributor
EUROND=	↓	-0.52/-0.41	BROKER
EURTND=	↑	-0.46/-0.36	BROKER
EURSND=	↓	-0.46/-0.36	KLIEM
EURSWD=	↑	-0.45/-0.35	CARL KLIEM
EUR2WD=	↑	-0.44/-0.35	CARL KLIEM
EUR3WD=	↓	-0.46/-0.36	KLIEM
EUR1MD=	↓	-0.46/-0.36	BROKER
EUR2MD=	↓	-0.43/-0.33	BROKER
EUR3MD=	↑	-0.38/-0.29	CARL KLIEM
EUR4MD=	↑	-0.36/-0.26	BROKER
EUR5MD=	↓	-0.35/-0.23	CARL KLIEM
EUR6MD=	↑	-0.29/-0.20	CARL KLIEM
EUR7MD=	↑	-0.26/-0.16	KLIEM
EUR8MD=	↓	-0.25/-0.13	CARL KLIEM
EUR9MD=	↓	-0.22/-0.12	BROKER
EUR10MD=	↓	-0.21/-0.10	CARL KLIEM
EUR11MD=	↑	-0.17/-0.08	CARL KLIEM
EUR1YD=	↓	-0.17/-0.06	CARL KLIEM
EUR2YD=	↑	-0.13/0.06	CM CIC
EUR3YD=	↓	-0.10/0.30	KLIEM
EUR4YD=	↓	-0.10/0.30	KLIEM
EUR5YD=	↓	0.10/0.50	KLIEM
EUR7YD=	↓	0.20/0.60	KLIEM
EUR10YD=	↓	0.50/0.90	KLIEM

24.10.2016 8:26:22 (GMT+1)

EU5YC6E=TTKL

Contributor: TULL PBON — Loc: LON — Time: 14:48 — Date: 14OCT16

Cap/Floor - OIS Discounting

Lognormal Vol	0	Currency	EUR
Norm Vol	39.34	Expiry	5Y
Basis Point Vol		Ref Rate	6M EURIBOR
Shifted Lognormal	Shift	Skew/Strike	ATM
		Ref	EURCAP=TTKL
Cap Premium		Related Data	
Floor Premium			
Straddle Premium	225.5		
ATM Strike	-0.051		

24.10.2016 8:24:33 (GMT+1)

Source: Author

The Normal Distribution Model (Bachelier Model)

- One solution is to apply a simple normal distribution model

$$dF_t = \sigma_N dW_t$$

$$F_t = F_0 + \sigma_N W_t.$$

$$F_T \sim N(F_0, \sigma_N^2 T)$$

$$c_N(T,K) = e^{-rT} E_T[\max(F_T - K, 0)]$$

$$c_N(T,K) = e^{-rT}\left[(F_t - K)N(d) + \sigma\sqrt{t}\,N'(d)\right] \qquad d = \frac{F_t - K}{\sigma_N \sqrt{t}}.$$

Shifted Lognormal Model (Displaced Diffusion)

- Another alternative is to shift the basic level

$$dF_t = d(F_t - \Theta) = \sigma_{DD}(F_t - \Theta)dW_t$$

$$F_t = \Theta + (F_0 - \Theta)\exp\left(\sigma_{DD}W_t - \frac{1}{2}\sigma_{DD}^2 t\right)$$

- Blacks (1976) formula can be applied

$$C_{DD}(T,K,F_t) = C_{B76}\left(T, K-\Theta, F_t-\Theta, \sigma_{impl}^{DD}(T, K-\Theta)\right)$$

$$P_{DD}(T,K,F_t) = P_{B76}\left(T, K-\Theta, F_t-\Theta, \sigma_{impl}^{DD}(T, K-\Theta)\right)$$

A Note on Binary Options

- A binary (cash) option pays just a fixed amount Q if it is exercised, for example a binary call

$$c_T = Q \times I\{F_T \geq K\}$$

$$c_0 = P(0,T)Q \times E_T[I\{F_T \geq K\}] = P(0,T)Q \times \Pr_T[F_T \geq K]$$

- Therefore, its valuation is quite simple in the normal and lognormal models

$$c_0^N = P(0,T)QN\left(\frac{F_0 - K}{\sigma_N^2 T}\right) \qquad c_0^{LN} = P(0,T)QN(d_2), \quad d_2 = \frac{\ln\left(F_0 / K\right) - \sigma_{LN}^2 T / 2}{\sigma_{LN}\sqrt{T}}$$

A Comparison of the Models

Category	Lognor-mal	Normal	Shifted LN
Interest rate	F>0	-∞ <F<∞	F>Θ (Θ<0)
Option price C/P	Black'76	own formula	Shifted Black'76
Volatility level	independent of interest rate	dependent on interest rate	independent of interest rate
Degree of reality	high until 2011, now partly unac-ceptable	unrealistically even deflec-tions up and down	realistic, but dynamic shift adjustments

Source: Author

Content

- ➢ Convexity, time, and quanto adjustments
- Short-rate and advanced interest rate models
- Volatility smiles
- Exotic options
- Alternative stochastic models
- Numerical methods for option pricing
- Credit derivatives

Exotic swaps

- Step-up swaps – increasing notional
- Amortizing swaps – decreasing notional
- Basis swaps – different reference rates
- Compounding swaps – the interest payments are compounded forward to the maturity date
- Libor-in-arrears swaps
- Constant maturity swaps – floats are swap rates in arrears with constant maturity
- Differential swaps – reference float is in a different currency than the notional (and payments)
- Equity swaps / equity return x fixed return
- Accrual swaps, cancelable swaps, cancelable compounding swaps, amortizing rate swaps, commodity swaps, volatility swaps,….

Real Life Example

- In 2003 the City of Prague has entered into a 10 year EUR/CZK cross currency swap with nominal 170 mil. EUR, fixed coupon in EUR and float coupon in CZK defined as fix – (IRS10 – IRS2)
- Different valuations estimated the initial market value at a loss between 190 to 280 mil. CZK. Most of the valuations did not use the convexity adjustment.

Convexity Adjustment

- In principle, we need to value $f(0)$ where the payoff $f(T)=s(T)$ is *N* year IRS swap rate quoted at time T.
- We have shown that if the numeraire $A(t)$ is the sum of values $P(t,T_i)$ of zero-coupon bonds paying 1 at the swap payment dates $T_1,\ldots,T_N$ then

$$s(0) = E_A[s(T)], \text{ where}$$

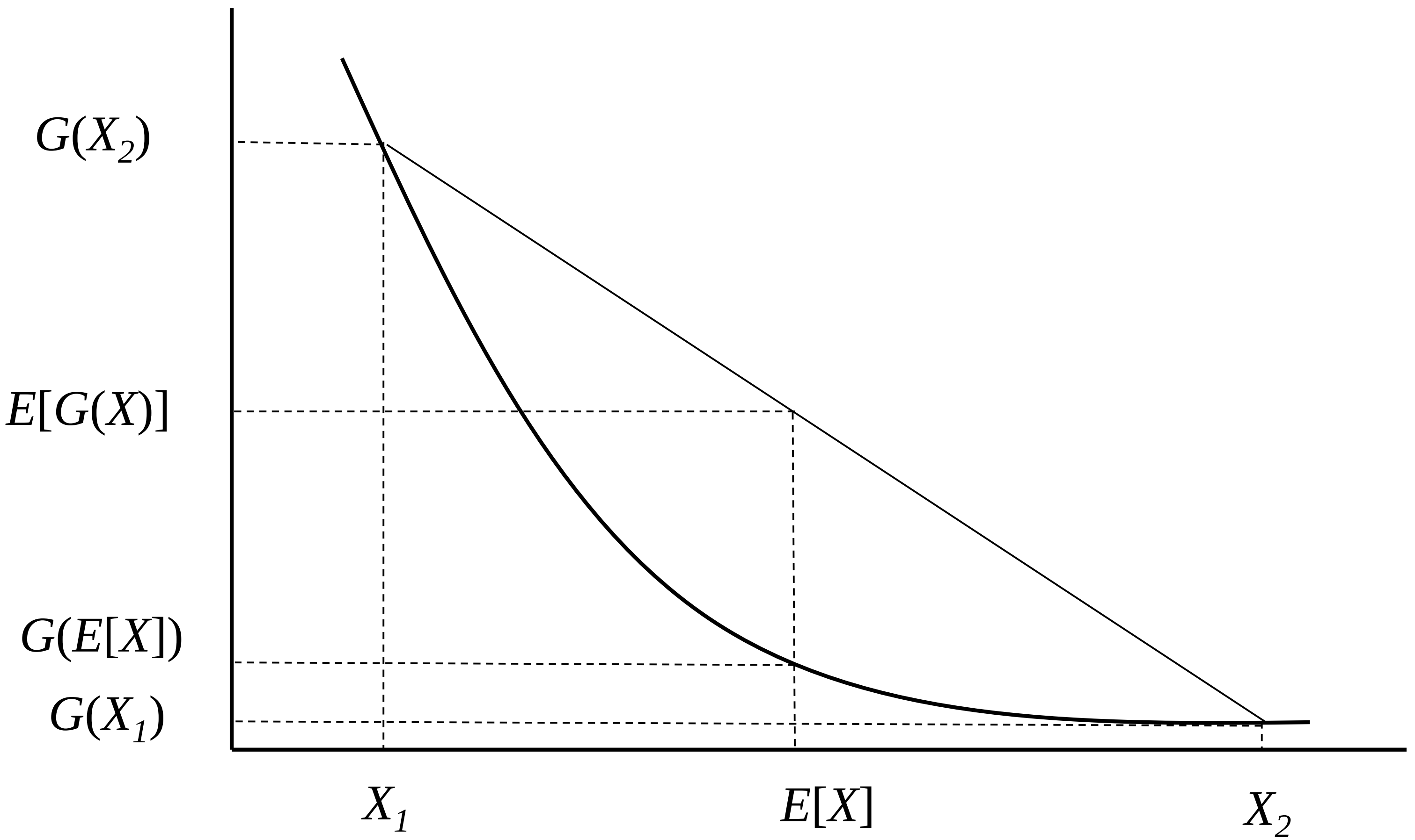
$$s(t) = \frac{P(t,T_0) - P(t,T_N)}{A(t)}$$

- But it is not correct to replace $s(T)$ with the forward rate in the normal risk-neutral world!!!
- En estimation of the difference between the expected value in the two measures yields a convexity adjustment
- The adjusted market value of the City of Prague swap is -244 mil. CZK

Convexity Adjustments in General

- If F is the maturity T forward price of an asset with spot price S then $F=E_T[S_T]$ w.r.t. $P(t,T)$ risk neutral measure, but not w.r.t. another measure
- If $R(t,T,T^*)$ denotes the forward interest rate then $R(0,T,T^*)=E_{T^*}[R(T,T,T^*)]$ w.r.t. $P(t,T^*)$ risk neutral measure, but not w.r.t. $P(t,T)$!!!
- **In general,** let an asset price be $B=G(y)$,or $y=G^{-1}(B)$
- If $B_F=E_T[B_T]$ is the maturity T forward price of B then we can define the forward rate $y_F=G^{-1}(B_F)$
- If G is nonlinear then $B_F = E_T[G(y_T)] <> G(E_T[y_T])$, i.e. $y_F <> E_T[y_T]$, and an adjustment is needed

Convexity adjustment



Source: Author

Jensen's inequality: If G is convex then

$$G\left(E[X]\right) \leq E[G(X)]$$

Convexity Adjustment Analytical Aproximation

- Expand $G(y_T)$ using a Taylor series at y_F up to the second order element and apply E_T to both sides of the expansion
- Approximate $G(y_T) \cong G(y_F) + (y_T - y_F)G'(y_F) + \frac{1}{2}(y_T - y_F)^2 G''(y_F)$
- To get

$$E_T[(y_T - y_F)^2] \approx \sigma_y{}^2 y_F{}^2 T$$

$$E_T[y_T] \approx y_F - \frac{1}{2} y_F^2 \sigma_y^2 T \frac{G''(y_F)}{G'(y_F)}$$

- Apply to swap rates in arrears approximated by YTM y of a corresponding B, i.e. derivatives corresponding to the duration and convexity
- Or to interest rates in arrears using $G(y)=1/(1+y(T^*-T))$

Change of Numeraire

- Sometimes we need to start with one numeraire g and change it to another numeraire h. The drift of a derivative f is then changed by

$$\alpha = \rho \sigma_f \sigma_w$$

- Where $w=h/g$ is the numeraire ratio and ρ the correlation between f and w
- Therefore, if α is a constant, then

$$E_h\left[f(T)\right] = E_g\left[f(T)\right]e^{\alpha T}$$

Change of Numeraire

$$\mu_f = r + \sum_{i=1}^{m} \sigma_{g,i}\sigma_{f,i} \qquad \mu'_f = r + \sum_{i=1}^{m} \sigma_{h,i}\sigma_{f,i}$$

$$\alpha_f = \mu'_f - \mu_f = \sum_{i=1}^{m} \left(\sigma_{h,i} - \sigma_{g,i}\right)\sigma_{f,i} = \sum_{i=1}^{m} \sigma_{w,i}\sigma_{f,i}$$

Using Ito's lemma applied to $\ln w = \ln h - \ln g$

$$df = \mu_f\, f dt + \sum_{i=1}^{m} \sigma_{f,i}\, f dz_i \qquad dw = \mu_w w dt + \sum_{i=1}^{m} \sigma_{w,i} w dz_i$$

$$\operatorname{cov}(df, dw) = E\left[\left(\sum_{i=1}^{m} \sigma_{f,i}\, f dz_i\right)\left(\sum_{j=1}^{m} \sigma_{w,j} w dz_j\right)\right] = \left(\sum_{i=1}^{m} \sigma_{f,i}\sigma_{w,i}\right) f w dt$$

$$\sum_{i=1}^{m} \sigma_{w,i}\sigma_{f,i} = \frac{\operatorname{cov}(df, dw)}{f w\, dt} = \rho\sigma_f\sigma_w$$

Source: Author

Timing Adjustments

- How do we calculate the value of a derivative with payoff = V_T paid at time $T^* > T$?
- We need $E_{T^*}[V_T]$, but we know $E_T[V_T]$ = forward price if V is a tradable asset
- The change of numeraire $W = P(t,T^*)/P(t,T)$ increases drift of V by $\alpha = \rho_{VW}\sigma_V\sigma_{W,}$, i.e. $E_{T^*}[V_T] = E_T[V_T]\, e^{\alpha T}$
- We may in fact express the adjustment in terms of $T \times T^*$ interest rate volatility and its correlation with V

Quantos

- The value of a financial instrument paid in a different (i.e. "wrong") currency, e.g. Nikkei index value paid in USD
- We would like to have $E_{USD}[V_T]$ expressed using $E_{YEN}[V_T] = V_F$, where V_T= Nikkei index value
- However, let us use two USD denominated numeraires $h= P_{USD}(t,T)$, $g= P_{YEN}(t,T)/S(\mathrm{t})$, and note that $E_g[V_T] = V_F$
- To get the adjustment look at the numeraire ratio $W= S(t)P_{USD}(t,T) / P_{YEN}(t,T)$ = forward exchange rate where $S(t)$ is the spot USD/YEN exchange rate
- In case of the Nikkei quanto, $T=1$, we need Nikkei volatility, 1Y USD/YEN forward volatility, and the correlation.

$$E_h[V_T] = E_g[V_T]e^{\rho_{VW}\sigma_V\sigma_W T}$$

Quantos - Example

- CME lists Nikkei 225 index futures settled in JPY and in USD. On February 13, 2012 the closing prices were:
  - Nikkei JPY contract = 8935
  - Nikkei USD contract = 8965
- Historical volatilities and correlations were estimated as:
  - Nikkei volatility = 20% p.a.
  - USD/JPY volatility = 12% p.a.
  - Correlation of Nikkei vs. USD/JPY returns = 35%
- According to the formula for quanto adjustment, the futures price of Nikkei settled in USD should be:
- $E[I_T] = F_0 e^{\rho \sigma_W \sigma_I T} = 8395 e^{0.35*0.12*0.2*(5/12)} = 8966$
- Which is close to the quoted price of 8965

Content

- ✓ Convexity, time, and quanto adjustments
- ➢ <u>Short-rate and advanced interest rate models</u>
- Volatility smiles
- Exotic options
- Alternative stochastic models
- Numerical methods for option pricing
- Credit derivatives

Stochastic Interest Rate Models

- The Standard Market Model uses the assumption that interest rates and bond prices are lognormally distributed at certain point in time in the future
- It does not describe the stochastic dynamics of the interest rates
- **Stochastic Interest Rate Models**
  1. **Short-Rate Models** – Model the instantaneous interest rate and use it to derive the implied movement of the term structure
     - Eqiuilibrium models (Vasicek model, CIR model, etc.)
     - Non-Arbitrage models (Ho-Lee model, Hull-White model, etc.)
     - One-factor vs. Multi-factor models
  2. **Term Structure Models** – Model the behavior of the whole interest rate term structure
     - Heath-Jarrow-Morton (HJM) model
     - Libor Market Model (LMM)

Stochastic Models of the Short Rate

- The Standard Market Models do not model evolution of interest rates in time
- Short rate r = instantaneous short rate
- The goal is to model $r(t)$ in the traditional risk-neutral world (numerair = MM account) and use it to obtain the dynamics of the full term-structure of interest rates
- One or more factors: $dr = m(r,t)dt + s(r,t)dz$

$$P(t,T) = \hat{E}\left[e^{-\int_t^T r(\tau)d\tau} \right] = \hat{E}\left[e^{-\bar{r}(T-t)} \right]$$

$$\bar{r}(t,T) = \frac{1}{T-t}\int_t^T r(\tau)d\tau$$

$$R(t,T) = -\frac{1}{T-t}\ln P(t,T)$$

Equilibrium Models

- The initial term structure corresponds to an equilibrium given by the model, not necessarily to the observed term structure
- The (Dothan) Rendelman and Bartter Model – geometric Brownian motion

$$dr = \mu r dt + \sigma r dz$$

- Simple, but does not capture the mean reversion that can be empirically observed
- The money market account value explodes
- Analytical tractability only partial, not an affine model

The Vasicek Model

$$dr = a(b - r)dt + \sigma dz$$

- The stochastic differential equation can be solved analytically.
- Apply the Ito formula to $G(r,t)=e^{at}r$ in to get $dg = abe^{at}dt + \sigma e^{at}dz$, and solve for $r(t)$ to obtain:

$$r(t) = f(t, x(t)) = e^{-at}r(0) + b(1 - e^{-at}) + \sigma e^{-at}x(t)$$

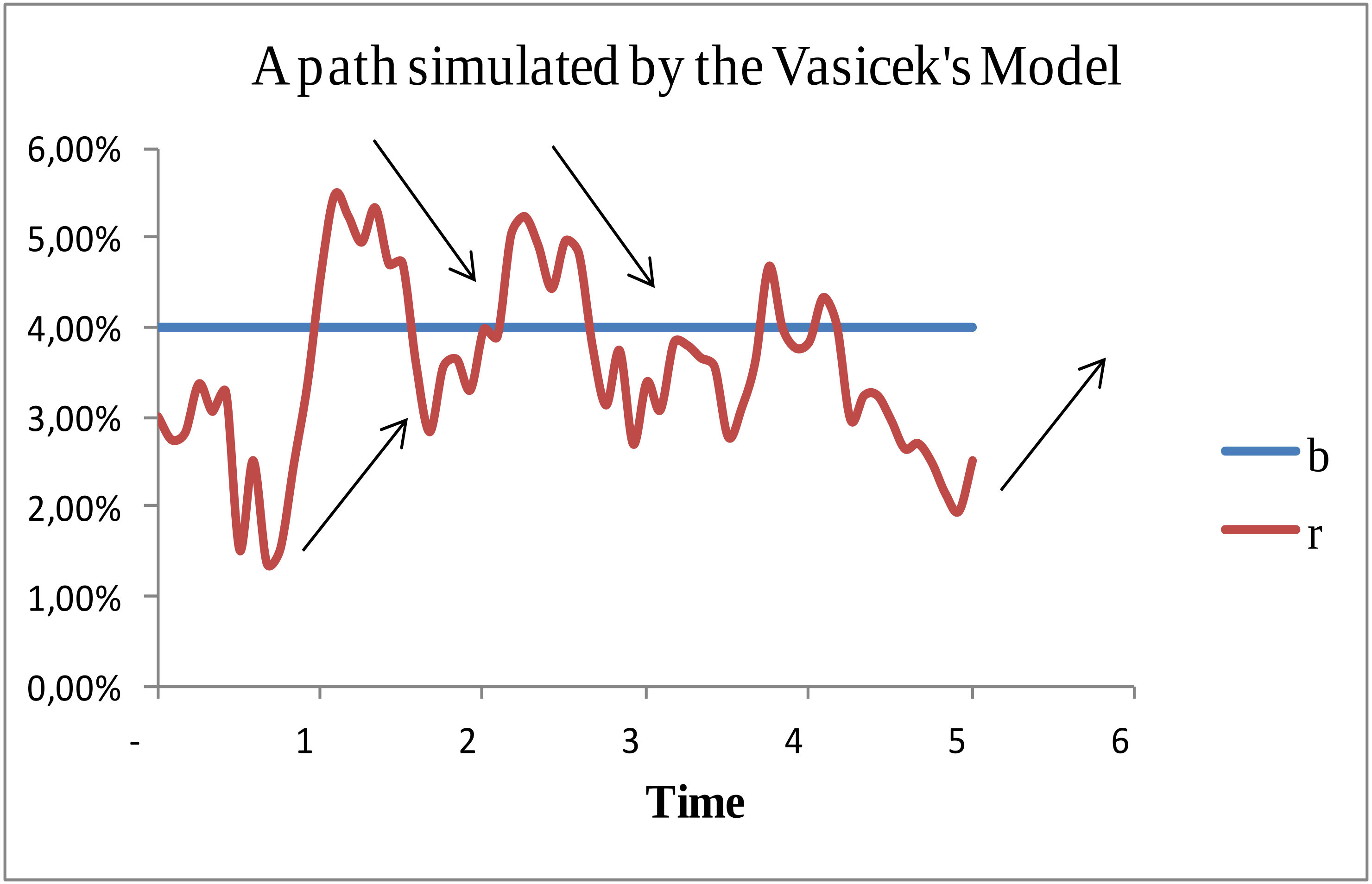
where

$$dx = e^{at}dz, \text{ i.e. } x(t) = \int_0^t e^{as}dz(s)$$ **is normally distributed**

Note that we may also analytically express $\bar{r}(t) = \frac{1}{t}\int_0^t r(s)ds$

$$\text{var}[x(t)] = \int_0^t e^{2as}ds = \frac{1}{2a}\left(e^{2at} - 1\right) \qquad \text{var}[r(t)] = \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right)$$

The Mean Reversion property



Source: Author

The Vasicek Model

- The corresponding (affine) term structure can be expressed analytically

$$R(t,T)=\alpha(t,T)+\beta(t,T)r(t) \quad \ldots \quad P(t,T)=A(t,T)e^{-B(t,T)r(t)}$$

$$\beta(t,T)=B(t,T)/\left(T-t\right)$$

$$\alpha(t,T)=-\left(\ln A(t,T)\right)/\left(T-t\right)$$

$$B(t,T)=\frac{1-e^{-a(T-t)}}{a}$$

$$A(t,T)=\exp\left(\frac{\left(B(t,T)-T+t\right)\left(a^2b-\sigma^2/2\right)}{a^2}-\frac{\sigma^2 B(t,T)^2}{4a}\right)$$

- Use the Ito formula to set up a PDE for $P(t,T)=f(t,r)$ and find f in the form

$$f(t,r)=A(t,T)e^{-B(t,T)r(t)}$$

Affine term-structure models PDE

Affine models are the models where: $R(t,T)=\alpha(t,T)+\beta(t,T)r(t)$

Proposition: The short rate model is affine if: $m(r,t)=\lambda(t)r+\eta(t)$

$$dr=m(r,t)dt+s(r,t)dz \qquad s^2(r,t)=\gamma(t)r+\delta$$

Proof:

assume $P(t,T)=A(t,T)e^{-B(t,T)r(t)}$

then $$dP=\left(A'e^{-Br}-AB're^{-Br}-ABe^{-Br}m+\frac{1}{2}AB^2s^2e^{-Br}\right)dt-ABe^{-Br}sdz$$

$$A'e^{-Br}-AB'e^{-Br}-ABe^{-Br}m+\frac{1}{2}AB^2s^2e^{-Br}=rAe^{-Br}$$

$$\left(\frac{A'}{A}-B\eta+\frac{1}{2}B^2\delta\right)+\left(-B'-B\lambda+\frac{1}{2}B^2\gamma-1\right)r=0$$

We get two ODE, that can be solved:

$$-B'-B\lambda+\frac{1}{2}B^2\gamma-1=0,$$

$$(\ln A)'-B\eta+\frac{1}{2}B^2\delta=0.$$

Boundary conditions: $A(T,T)=1$, $B(T,T)=0$

Vasicek Model

$$\lambda = -a \quad \gamma = 0 \qquad -B' + aB = 1$$

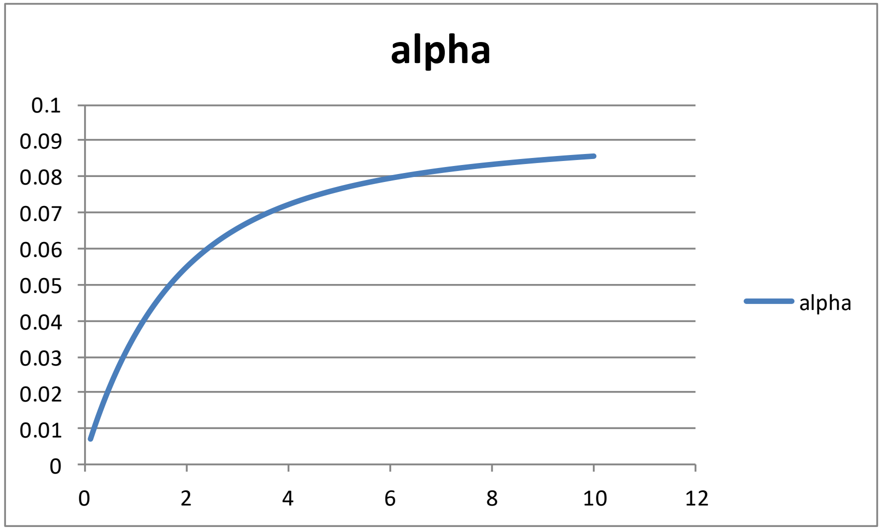
$$\eta = ab \quad \delta = \sigma^2 \qquad (\ln A)' = abB - \frac{1}{2}\sigma^2 B^2$$

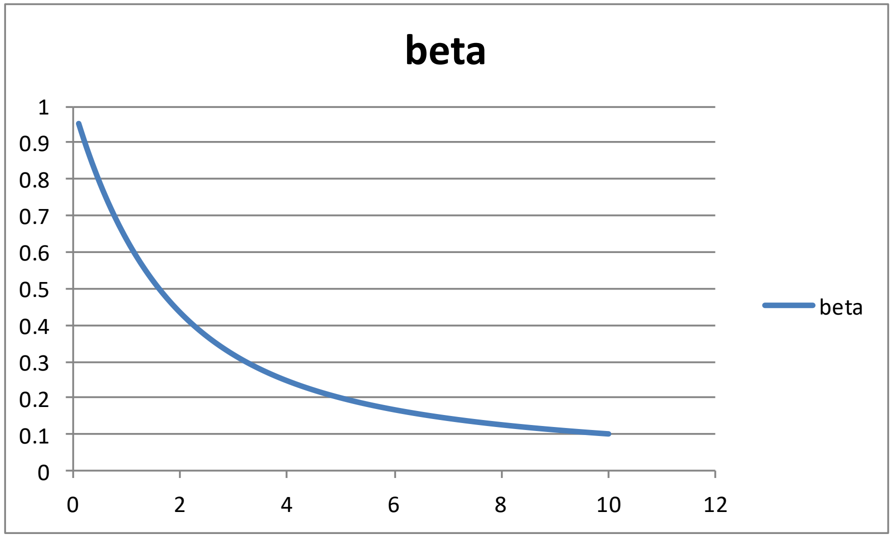
$$B(t,T) = \frac{1 - e^{-a(T-t)}}{a}$$

$$A(t,T) = \exp\left(\frac{\left(B(t,T) - T + t\right)\left(a^2 b - \sigma^2/2\right)}{a^2} - \frac{\sigma^2 B(t,T)^2}{4a} \right)$$

$$R(t,T) = \alpha(t,T) + \beta(t,T) r(t)$$

$$\alpha(t,T) = -\left(\ln A(t,T)\right) / \left(T - t\right) \qquad \beta(t,T) = B(t,T) / \left(T - t\right)$$





Source: Author

Vasicek Model - Calibration

- The expression for $P(t,T)$ can be used to calibrate the model to the observed interest rate term structure
- σ can be estimated from historical interest rates
- a and b need to be estimated via calibration
- For two maturities the model can be fitted exactly
- For more than two maturities, we minimize the sum of squared errors between the implied and observed interest rates

$$SSE(a,b)=\sum_i\left(R^M(0,T_i) - R^{Vas}(0,T_i;a,b,\sigma)\right)^2$$

- Where $R^M(0,T_i)$ is the market observed interest rate and $R^{Vas}(0,T_i;a,b,\sigma)$ is the interest rate implied by the model

Vasicek Model – Calibration Example

Vasicek model parameters		
r0	**0.07%**	initial instantaneous interest rate
a	**44.57%**	mean reversion parameter
b	**3.13%**	long-term interest rate
sigma	**1.00%**	annual volatility

$$A(t,T)=\exp\left(\frac{\left(B(t,T)-T+t\right)\left(a^2b-\sigma^2/2\right)}{a^2}-\frac{\sigma^2B(t,T)^2}{4a}\right)$$

$$B(t,T)=\frac{1-e^{-a(T-t)}}{a}$$

$$\alpha(t,T)=-\left(\ln A(t,T)\right)/\left(T-t\right)$$

$$\beta(t,T)=B(t,T)/\left(T-t\right)$$

$$R(t,T)=\alpha(t,T)+\beta(t,T)r(t)$$

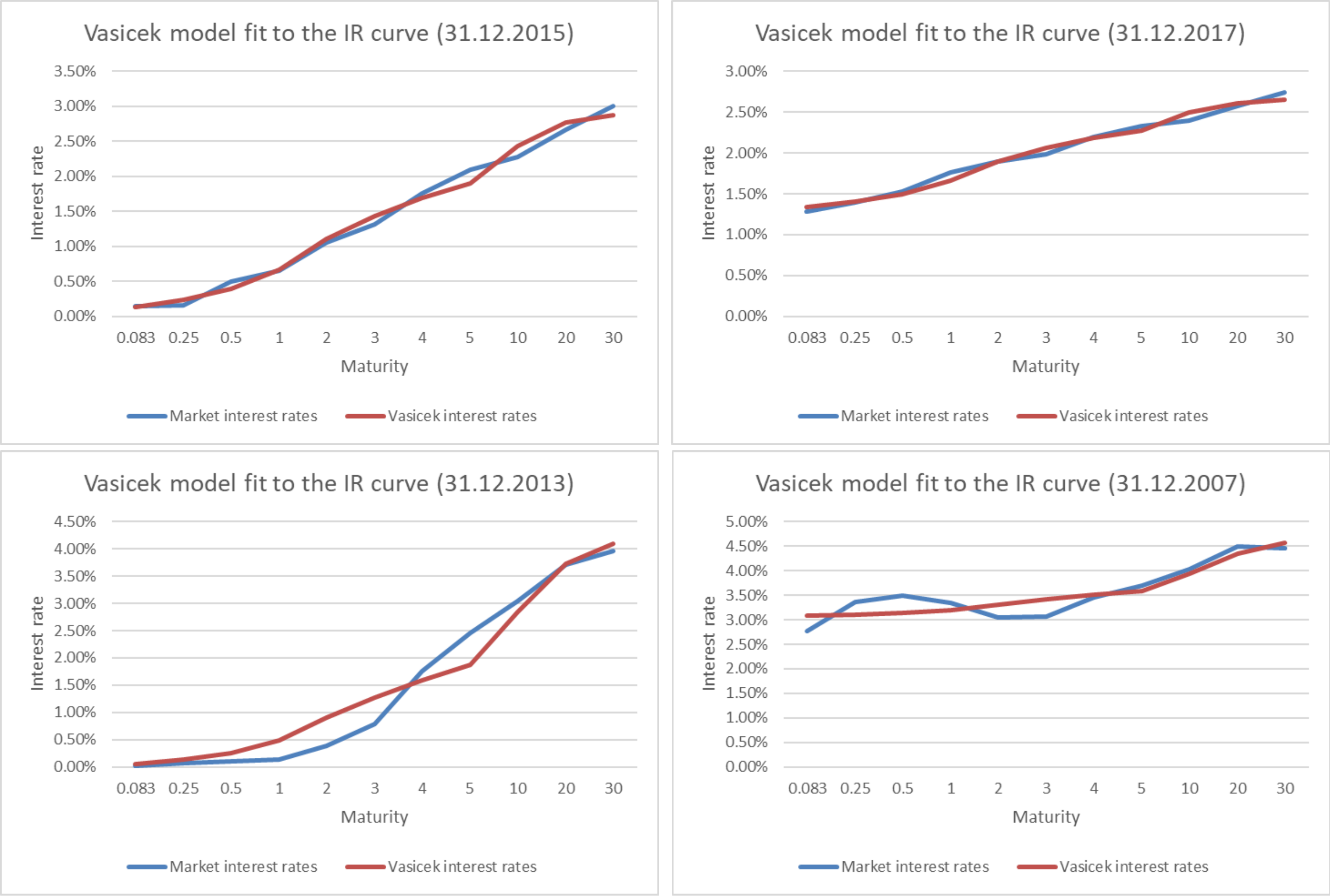
Interest rate term structure / **Calibration**

T	R	A(t,T)	B(t,T)	alfa(t,T)	beta(t,T)	R(t,T)	Diff	Diff^2
0.083	**0.14%**	0.999952	0.081805	0.000574	0.981655	**0.13%**	0.01%	1.5708E-08
0.25	**0.16%**	0.99958	0.236574	0.001679	0.946295	**0.24%**	-0.08%	5.6937E-07
0.5	**0.49%**	0.998383	0.448201	0.003237	0.896401	**0.39%**	0.10%	1.0466E-06
1	**0.65%**	0.993989	0.806858	0.006029	0.806858	**0.66%**	-0.01%	1.1068E-08
2	**1.06%**	0.979134	1.323528	0.010543	0.661764	**1.10%**	-0.04%	1.7305E-07
3	**1.31%**	0.958962	1.654376	0.013968	0.551459	**1.44%**	-0.13%	1.5918E-06
4	**1.76%**	0.935758	1.866234	0.0166	0.466558	**1.69%**	0.07%	4.4506E-07
5	**2.09%**	0.91097	2.001896	0.018649	0.400379	**1.89%**	0.20%	3.861E-06
10	**2.27%**	0.785258	2.217432	0.024174	0.221743	**2.43%**	-0.16%	2.6653E-06
20	**2.67%**	0.576248	2.243137	0.027561	0.112157	**2.76%**	-0.09%	8.8538E-07
30	**3.01%**	0.422539	2.243435	0.028716	0.074781	**2.88%**	0.13%	1.771E-06
							sum(diff^2)	**1.3035E-05**

Source: Author

1. Fill the initial parameters (r0, a, b, sigma), maturites (T) and market rates (R)
2. Compute the values of A(t,T), B(t,T), alfa(t,T), beta(t,T) and R(t,T)
3. Use Solver to find parameters (r0, a, b) that minimize sum{[R-R(t,T)]^2}

Vasicek Model – Calibration Results



Source: Author

- Vasicek model is unable to accurately fit all possible shapes of the IR curve

Vasicek Model

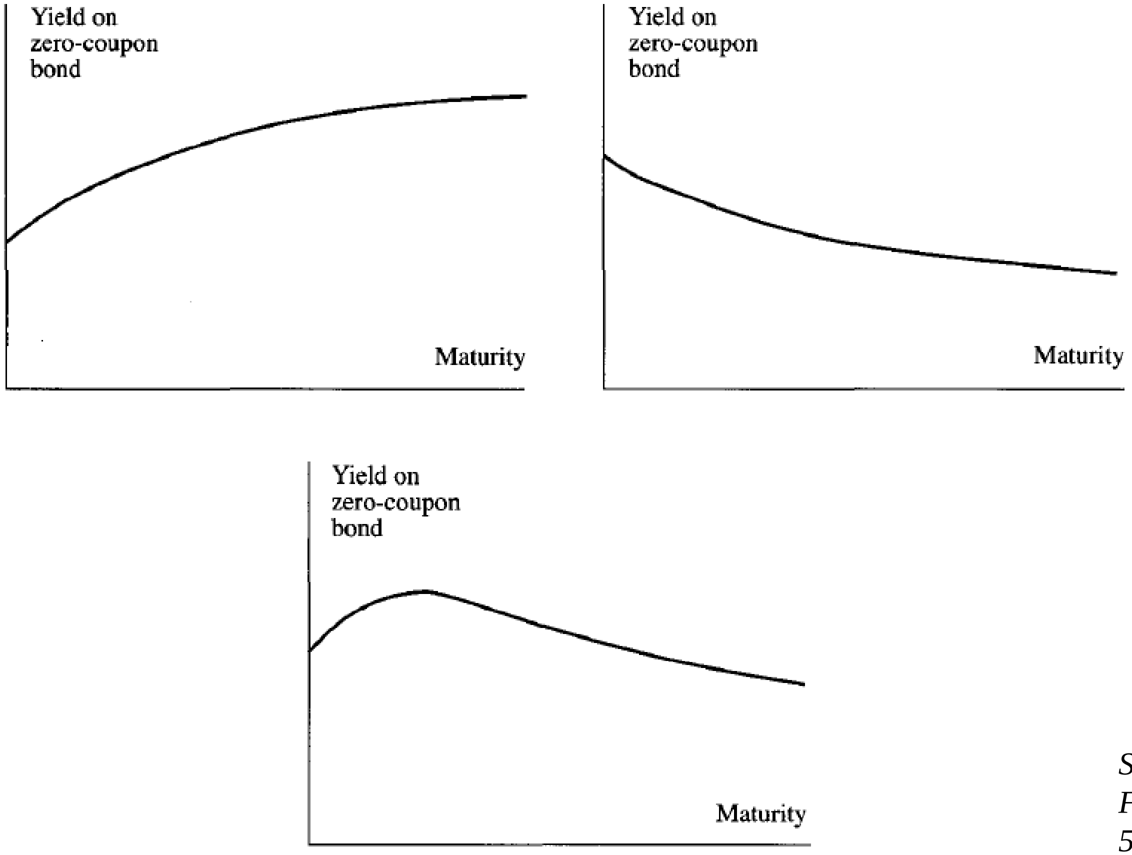


Figure 23.2 Possible shapes of term structure when the Vasicek model is used

Source: John Hull, Options, Futures, and Other Derivatives, 5th edition

Limited flexibility to fit the initial term structure!

Valuation of zero coupon bond Options in the Vasicek Model

- For a call option with strike K, maturing at T, on a zero coupon bond maturing at T^* with principal L we may obtain

$$c_0 = E\left[\exp\left(-\int_0^T r(s)ds\right) f_{\text{payoff}}(r(T))\right] = E\left[e^{-\bar{r}(T)T} f_{\text{payoff}}(r(T))\right]$$

$$c_0 = LP(0,T^*)N(h) - KP(0,T)N(h-\sigma_P)$$

$$h = \frac{1}{\sigma_P}\ln\frac{LP(0,T^*)}{P(0,T)K} + \frac{\sigma_P}{2} \qquad \sigma_P = \frac{\sigma}{a}\left(1-e^{-a(T^*-T)}\right)\sqrt{\frac{1-e^{-2aT}}{2a}}$$

- We need to use that $r(T)$ and the $\bar{r}(T)$ have bivariate normal distribution with a covariance that can be derived (Jamshidian); then get the expected value
- The PDE for $c(t,r)$ is the same as for $P=f(t,r)$ but there is a different boundary condition $c(T,r)=(f(T,r)-K)^+$

Valuation of caps and floors in the Vasicek Model

- Lets consider a caplet on the interest rate $R_M(T,T^*)$, expressed in MM compounding, exercised at time T^*, with a fixed exercise rate R_K
- The payoff of the caplet on principal L, discounted to T is:

$$\frac{L\delta(R_M - R_K)^+}{1 + R_M\delta} = \left(L - \frac{L(1 + R_K\delta)}{1 + R_M\delta}\right)^+ = \big(L - L(1 + R_K\delta)P(T,T^*)\big)^+$$

- Where δ is the time factor from T to T^*
- The caplet can thus be valued as a European put option on the zero coupon bond $P(T,T^*)$, multiplied by face value $L(1 + R_K\delta)$, with the strike price L
- Similarly, floorlet can be valued as a European call option

Cap valuation – Example (1)

- Lets assume we want to value cap on 100 million USD with 5.5 years to maturity, semi-annual payments and strike price equal to 3% p.a., with the valuation done the end of 2015.
- We first fit the parameters of the Vasicek model to the interest rate curve observed on 31.12.2015, we get:
- $a = 44.57\%$, $b = 3.13\%$, $\sigma = 1.00\%$ and $r_0 = 0.07\%$
- We can then value individual caplets as put options on zero coupon bonds $P(T, T^*)$, multiplied by face value $L(1 + R_K\delta)$ and with a strike price L
- The value p_0 of each caplet can be computed as:

$$p_0 = LP(0,T)N(-h + \sigma_P) - L(1 + R_K\delta)P(0,T^*)N(-h)$$

$$h = \frac{1}{\sigma_P}\ln\frac{L(1 + R_K\delta)P(0,T^*)}{P(0,T)L} + \frac{\sigma_P}{2} \qquad \sigma_P = \frac{\sigma}{a}\left(1 - e^{-a(T^*-T)}\right)\sqrt{\frac{1 - e^{-2aT}}{2a}}$$

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)}$$

Cap valuation – Example (2)

- The value of the cap is then given as the value of all of the caplets:

T	T*	A(0,T)	B(0,T)	P(0,T)	P(0,T*)	P(T,T*)	sigmaP	h	p
0.5	1	0.998383	0.448201	0.998063	0.993417	0.995344	0.002847	3.59208039	11.6017235
1	1.5	0.993989	0.806859	0.993417	0.986641	0.993179	0.003646	2.20817939	1745.35668
1.5	2	0.987412	1.093864	0.986641	0.978209	0.991454	0.004076	1.54897371	10615.2945
2	2.5	0.979134	1.32353	0.978209	0.968503	0.990078	0.00433	1.1378589	27135.3698
2.5	3	0.969546	1.507313	0.968503	0.95783	0.98898	0.004484	0.85126904	47934.3092
3	3.5	0.958962	1.654379	0.95783	0.946434	0.988103	0.00458	0.63978848	69692.5572
3.5	4	0.947633	1.772065	0.946434	0.934511	0.987402	0.004641	0.47866391	90229.0028
4	4.5	0.935758	1.866238	0.934511	0.922215	0.986842	0.004679	0.35356593	108400.322
4.5	5	0.923495	1.941598	0.922215	0.909668	0.986395	0.004704	0.25531928	123771.462
5	5.5	0.91097	2.001902	0.909668	0.896966	0.986037	0.004719	0.17761254	136329.271
5.5		0.898281	2.050158	0.896966					
								Cap	**615864.547**

Source: Author

$$p_0 = LP(0,T)N(-h+\sigma_P) - L(1+R_K\delta)P(0,T^*)N(-h)$$

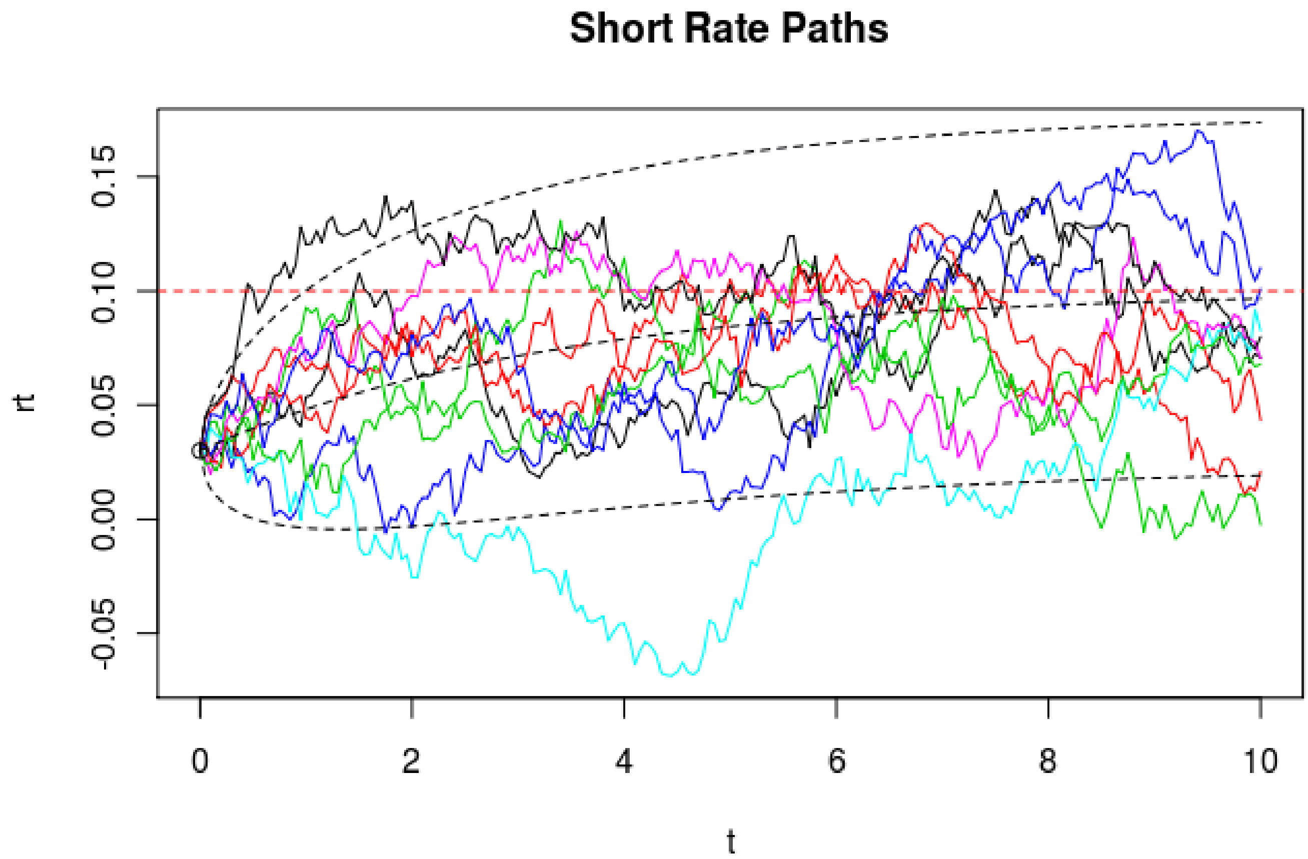
$$h = \frac{1}{\sigma_P}\ln\frac{L(1+R_K\delta)P(0,T^*)}{P(0,T)L} + \frac{\sigma_P}{2} \qquad \sigma_P = \frac{\sigma}{a}\left(1-e^{-a(T^*-T)}\right)\sqrt{\frac{1-e^{-2aT}}{2a}}$$

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)}$$

Valuation of fixed-coupon Bonds in the Vasicek Model

- European call and put options on fixed-coupon bonds can be valued with the Vasicek model by using the Jamshidian‘s trick
- Value of bond Q at time T is given by a series of discounted cash-flows $C_1, \ldots, C_n$ that can be represented as a weighted sum of zero-coupon bonds $P(T, T_1), \ldots, P(T, T_n)$, each depending monotically on the short rate $r(T)$
- Therefore, considering a time T European call option on a fixed coupon bond $Q = Q(r(T))$ with strike price K, there is a rate r^* so that $Q(r^*) = K$, and the call option will be exercised only if $r(T) < r^*$
- Payoff on the fixed coupon bond call option $\left(Q\left(r(T)\right) - K\right)^+$ can then be represented as a weighted sum of payoffs on the zero coupon bond call options $C_i\ (P(T, T_i) - K_i)^+$ with strikes $K_1, \ldots, K_n$ being the zero coupon bond values corresponding to r^* and T_i

Vasicek process simulation - Illustration



Source: https://www.r-bloggers.com/fun-with-the-vasicek-interest-rate-model/

We can see the tendency of the simulations to mean-revert towards the long-term level (equal to 0.1 in this case)

The Cox, Ross, Ingersoll Model

$$dr = a(b - r)dt + \sigma\sqrt{r}dz$$

- Modeled interest rates always non-negative (might be negative in the Vasicek model), provided $2ab > \sigma^2$
- $r(t)$ cannot be expressed analytically as in Vasicek model, but its distribution yes - non-central chi-squared
- $P(t,T)$ has an analytical solution – it is an affine model
- Options on bonds valued by formulas involving integral of the non-central chi-squared distribution

Non-Arbitrage Models

- Allow to fit the initial term-structure of interest rates (no instant arbitrage) which reflects the expected development of the short rates
- **The Ho-Lee model** – analytically tractable
- Gives the classical futures convexity adjustment

$$dr = \theta(t)dt + \sigma dz$$

$$\theta(t) = F'(0,t) + \sigma^2 t$$

F(0,t) … the forward rate

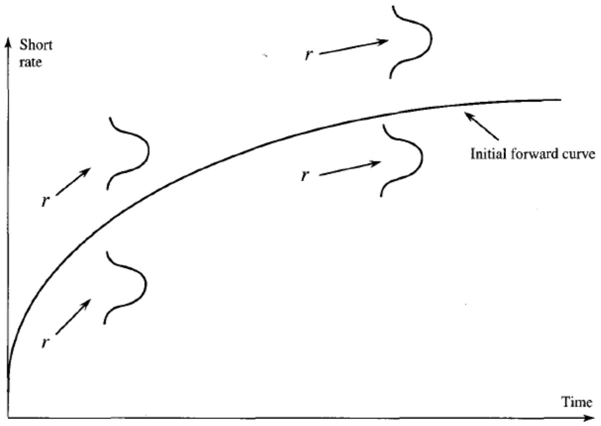


Figure 23.3 The Ho–Lee model

Source: John Hull, Options, Futures, and Other Derivatives, 5th edition

Ho-Lee model – the formula for θ

- The rate $r(t) = r(0) + \int_0^t \theta(s)ds + \sigma z(t)$ is normally distributed, affine model, similar option valuation formulas as for the Vasicek model

$$-B' = 1,$$

$$(\ln A)' - B\theta + \frac{1}{2}B^2\sigma^2 = 0.$$

$$B(t,T) = T - t$$

$$\ln A(t,T) = -\int_t^T (T-s)\theta(s)ds + \frac{1}{6}\sigma^2(T-t)^3$$

$$P(t,T) = A(t,T)e^{-(T-t)r(t)}$$

Take the log, $t = 0$

$$\ln P(0,T) + T \cdot r(0) = -\int_0^T (T-s)\theta(s)ds + \frac{1}{6}\sigma^2 T^3$$

Differentiate w.r.t. T

Result:

$$\theta(t) = F'(0,t) + \sigma^2 t$$

$$\frac{\partial^2}{\partial T^2}\ln P(0,T) = -\theta(T) + T\sigma^2$$

And note that

$$F(t,T) = \lim_{T_2 \to T} f(t,T,T_2) = -\frac{\partial}{\partial T}\ln P(t,T)$$

$$f(t,T_1,T_2) = \frac{\ln P(t,T_1) - \ln P(t,T_2)}{T_2 - T_1}$$

STIR Futures Convexity Adjustment

- The classical STIR convexity adjustment is proved in the context of Ho-lee model
- The futures rate is a martingale with respect to the traditional r.n. measure $F(0,T_1,T_2)=E\left[F(T_1,T_1,T_2)\right]$, but this is not the case of the forward rate

$$dP(t,T)=r(t)P(t,T)dt-(T-t)\sigma P(t,T)dz$$

by Ito's lemma

$$df=\sigma^2\frac{(T_2-t)^2-(T_1-t)^2}{2(T_2-T_1)}dt+\sigma dz$$

From the proces for *r* we derive proces for *P* and from it the proces for *f*

$$E\left[f(T_1,T_1,T_2)\right]-f(0,T_1,T_2)=\int_0^{T_1}\sigma^2\frac{(T_2-t)^2-(T_1-t)^2}{2(T_2-T_1)}dt=$$

Drift until maturity for the forward rate

$$=\frac{\sigma^2}{6(T_2-T_1)}\left[(T_1-t)^3-(T_2-t)^3\right]_0^{T_1}=\frac{\sigma^2}{2}T_1T_2.$$

Moreover $F(T_1,T_1,T_2)=f(T_1,T_1,T_2)$ so $F(0,T_1,T_2)-f(0,T_1,T_2)=\frac{\sigma^2}{2}T_1T_2.$

The Hull-White Model

- One Factor – generalization of the Vasicek model -again analytically tractable

$$dr = \left(\theta(t) - ar\right)dt + \sigma dz \qquad \theta(t) = F'(0,t) + aF(0,t) + \frac{\sigma^2}{2a}(1 - e^{-2at})$$

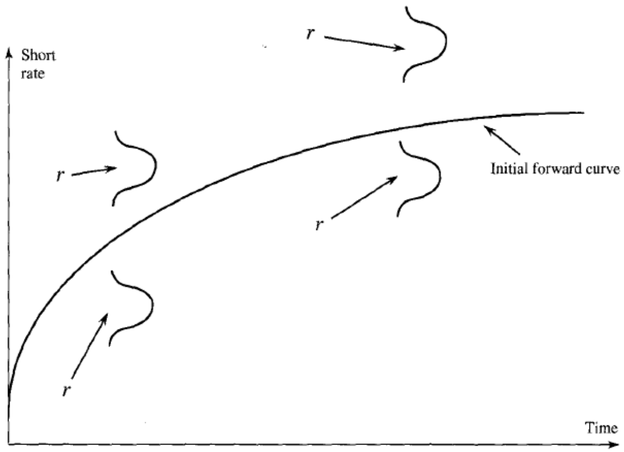
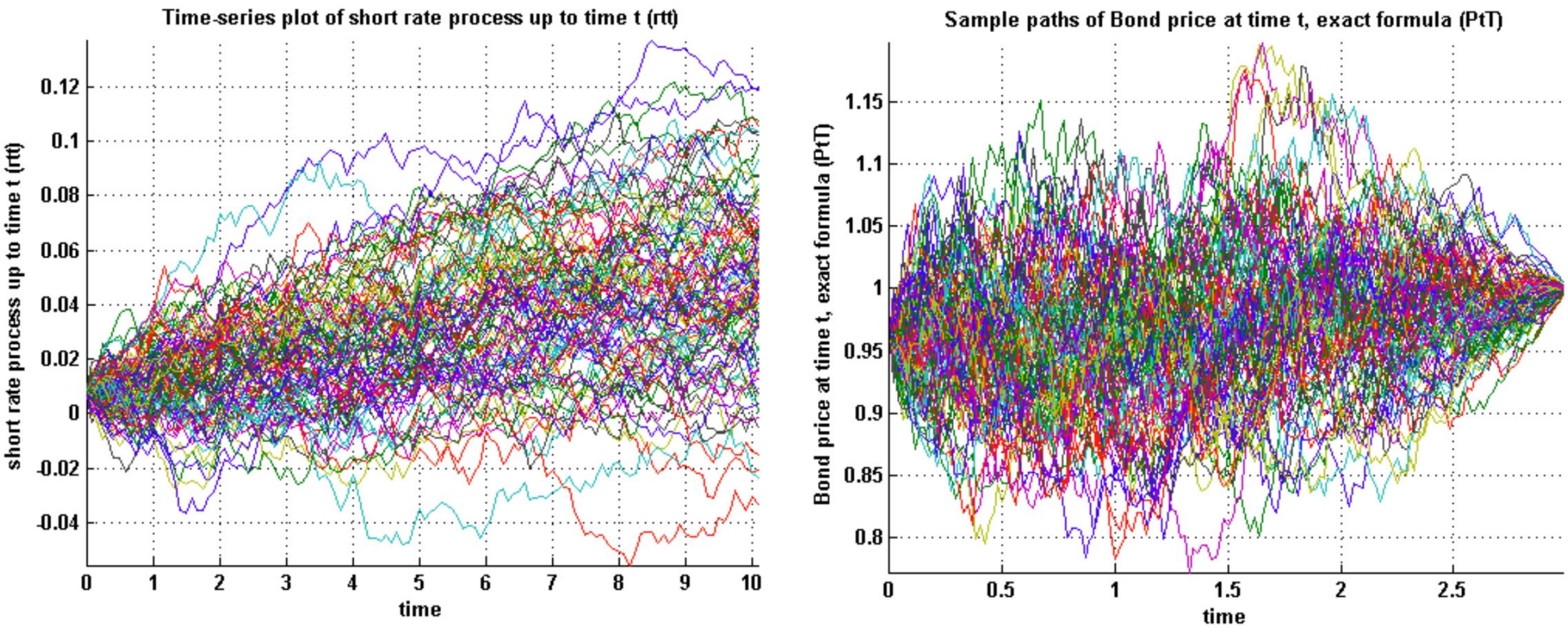


Figure 23.4 The Hull–White model

Source: John Hull, Options, Futures, and Other Derivatives, 5th edition

In the one-factor models the price of a bond depends just on r(t)
Two – factor Model … the reversion level given also by another process

Hull-White process simulation



Source: http://www.thetaris.com/wiki/Hull-White_model

Simulations of the Hull-White process can be used for the valuation of more complex options for which no analytical formulas exist

Other no-arbitrage models

- **No-Arbitrage CIR model**

$$dr = (\theta(t) - ar)dt + \sigma\sqrt{r}dz$$

- Problem: No analytical solutions for $\theta(t)$
- Analytical solution exists if we set volatility according to $\sigma(t) = \sqrt{\theta(t)/\delta}$ where $\delta > 1/2$ (Jamshidian, 1995)

- **Black-Karasinski model**
- Generalization of exponential-Vasicek model in which $y = \ln(r)$ follows the Vasicek model
- The B-K model adds time-dependent coefficients:

$$dy = (\theta(t) - a(t)y)dt + \sigma(t)dz$$

- Problem: No analytical solutions, Money Market explodes

Short-rate models – Summary

<u>Equilibrium models</u> (do not fit the zero-curve perfectly)	**<u>No-Arbitrage models</u>** (do fit the zero-curve perfectly)	**Characteristics**
Rendelman-Bartter	Ho-Lee model	No mean reversion Gaussian distribution Rate may be negative Analyticaly tractable
Vasicek model	Hull-White model	Has mean reversion Gaussian distribution Rate may be negative Analyticaly tractable
CIR model	No-Arbitrage CIR	Has mean reversion Non-Gaussian Rate can not be negative* Partically tractable
Exponential Vasicek	Black-Karasinsky	Has mean reversion Non-Gaussian Rate can never be negative Not tractable

*Rate may become negative due to discretization

Source: Author

Two-factor models

- In the one-factor affine models, whole term structure depends on the short rate $r(t)$ through the equation $R(t,T) = \alpha(t,T) + \beta(t,T)r(t)$
- Correlation between two rates $R(t,T_1)$ and $R(t,T_2)$ is thus always 1, which is unrealistic, and it becomes particularly problematic when pricing derivatives with payoffs depending on several interest rates
- The issue can be solved with two-factor affine models:
- $R(t,T) = \alpha(t,T) + \beta_1(t,T)x_1(t) + \beta_2(t,T)x_2(t)$
- $x_1(t)$ and $x_2(t)$ represent the sources of uncertainity, such as the short-rate and the long-rate
- All of the short-rate models can be extended into two-factor or multi-factor versions
- One-Factor models model the shift of the yield-curve, two-factor models add the slope, and three-factor models the curvature
- Brigo and Mercurio (2006) show that the first two components explain around 90% of the variations of the yield curve, while the first three components explain 95-99% of the variation

Two-factor Vasicek Model

- The **Two-Factor Vasicek Model** looks like:
- $r = x_1 + x_2$
- $dx_1 = k_1(\theta_1 - x_1)dt + \sigma_1 dz_1$
- $dx_2 = k_2(\theta_2 - x_2)dt + \sigma_2 dz_2$
- With instantaneous correlation $dz_1 dz_2 = \rho dt$
- The model can be extended with deterministic shift:
- $r = x_1 + x_2 + \varphi(t)$
- In order to exactly fit the initial zero-coupon term-structure (making it a two-factor version of the Hull-White Model)

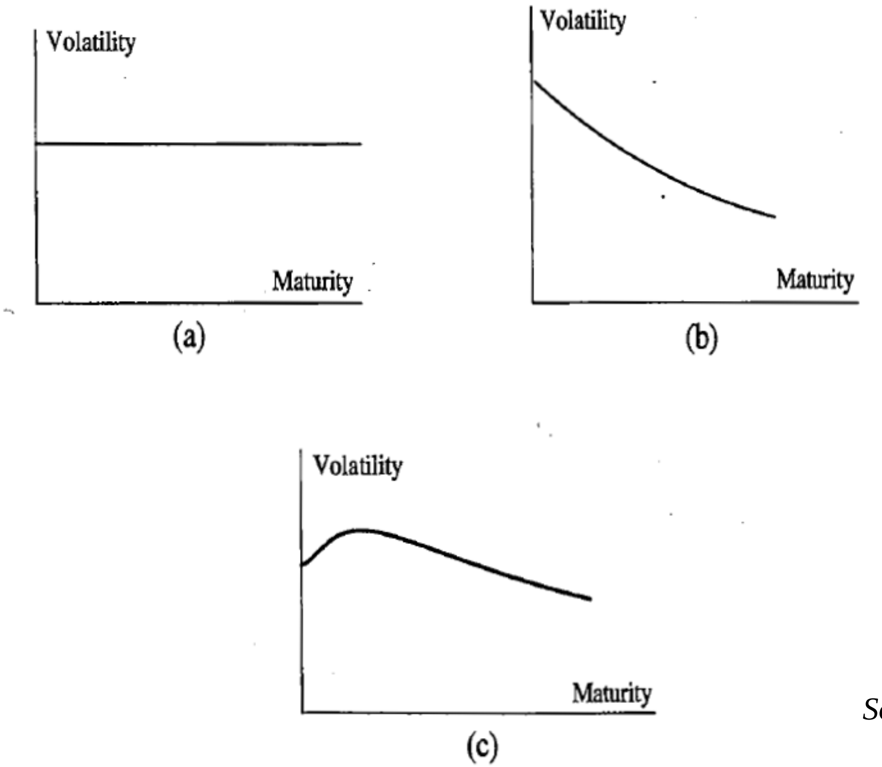
Options on fixed-coupon bonds

- In the one-factor models values of zero-coupon bonds decline with rising short rate
- Consequently, a fix coupon bond option can be decomposed into a series of zero coupon bond options
- Calculate the critical short rate r^* when the option on the bond is exercised
- Calculate strike prices of the individual zero coupon bond options corresponding to r^*
- Sum up the values of the zero coupon bond options
- Remark: the procedure can be also used to value swaptions

Volatility Structures

- The short rate models determine different pattern of forward rate volatilities

Figure 30.5 Volatility of 3-month forward rate as a function of maturity for (a) the Ho–Lee model, (b) the Hull–White one-factor model, and (c) the Hull–White two-factor model (when parameters are chosen appropriately).



Source: Author

HJM (Heath, Jarrow, and Morton) Term-structure Models

- Allow more flexibility in choosing the volatility term structure, one or more factors

$$dF(t,T) = m(t,T)dt + s(t,T)dz(t), \text{ for } t \le T \qquad F(0,T) = F_M(0,T)$$

- Starts with a process for the discounted bond price with the standard risk neutral measure

$$dP(t,T) = r(t)P(t,T)dt + v(t,T)P(t,T)dz$$

- Arrives at a process for forward rates

Apply Itoo‘s Lemma to *f*

$$f(t,T_1,T_2) = \frac{\ln P(t,T_1) - \ln P(t,T_2)}{T_2 - T_1}$$

$$df(t,T_1,T_2) = \frac{v(t,T_2)^2 - v(t,T_1)^2}{2(T_2 - T_1)} dt + \frac{v(t,T_1) - v(t,T_2)}{T_2 - T_1} dz$$

HJM Interest rate Model

- …and for instantaneous forward rates

$$\frac{1}{2}\frac{\partial v(t,T)^2}{\partial T} = v(t,T)v_T(t,T)$$

$v_T(t,T)$ is the derivative of the zero bond volatility curve

$$dF(t,T) = v(t,T)v_T(t,T)dt - v_T(t,T)dz$$

- Consequently, if we model the forward rates by

$$v(t,T) = -\int_t^T s(t,\tau)d\tau$$

Note that $v(T,T) = 0$

- Then the following HJM no-arbitrage condition must be satisfied

$$m(t,T) = s(t,T)\int_t^T s(t,\tau)d\tau$$

To model $F(t,T)$ we just need to estimate $s(t)$ (from historical data), then we can express the processes $r(t)=F(t,t)$ ..depends also on $F(0,t)$.. and $P(t,T)$ as well as the corresponding derivatives – fits the current term structure of interest rates as well as the volatility structure, but in general the Process is not Markov

HJM non-Markovian behavior

- The short-rate process in the HJM framework is non-Markovian
- i.e. as $r(t) = F(t,t)$, it holds that:
- $F(t,t,) = F(0,t) + \int_0^t dF(\tau,t)$
- Replacing $dF(\tau,t)$ with the HJM process we get:
- $r(t) = F(0,t) + \int_0^t v(\tau,t)v_t(\tau,t)d\tau - \int_0^t v_t(\tau,t)dz(\tau)$
- The third terms depends on the path of z from 0 to t
- In addition the second term may also become time-dependent if $v(\tau,t)$ is stochastic
- The non-Markovian behavior makes binomial trees non-recombining

The Libor Market Model

- The instantaneous forward rates are not directly observable on the market – difficult calibration of the HJM model
- Libor market model is expressed in terms of forward "Libor" rates

$F_k(t)$: Forward rate between times t_k and t_{k+1} as seen at time t, expressed with a compounding period of δ_k

$m(t)$: Index for the next reset date at time t; this means that $m(t)$ is the smallest integer such that $t \leqslant t_{m(t)}$

$\zeta_k(t)$: Volatility of $F_k(t)$ at time t

$v_k(t)$: Volatility of the zero-coupon bond price, $P(t, t_k)$, at time t

The Libor Market Model

- Forward rates $F_k(t)$ are martingales With respect to the $P(t,t_{k+1})$ risk neutral measure

$$dF_k(t) = \zeta_k(t) F_k(t) dz \qquad F_k(t) = \frac{1}{\delta_k} \frac{P(t,t_k) - P(t,t_{k+1})}{P(t,t_{k+1})}$$

- But we need to change the numeraire to a “rolling CD” where the cash is always reinvested into „$P(t_k,t_{k+1})$“. With respect to the changed measure

$$h(t) = P(t,t_m) h(t_{m-1}) P(t_{m-1},t_m)^{-1}$$

$$dF_k(t) = \zeta_k(t)\left(v_{m(t)}(t) - v_{k+1}(t)\right) F_k(t) dt + \zeta_k(t) F_k(t) dz$$

LMM change of numeraire

- We need to make the process of $F_k(t)$ risk-neutral with respect to the rolling CD account:

$$h(t) = P(t, t_m) h(t_{m-1}) P(t_{m-1}, t_m)^{-1}$$

- We will apply the change of numeraire technique to change the numeraire $P(t, t_{k+1})$ to $P(t, t_{m(t)})$
- The change of drift will be $\rho \sigma_w \sigma_f$, where $w = P(t, t_{m(t)})/P(t, t_{k+1})$ is the numeraire ratio, $f = F_k(t)$, and ρ is the instantaneous correlation between w and f
- If $v_k(t)$ denotes the volatility of $P(t, t_k)$ and:
- $dP(t, t_m) = (\dots)dt + v_m P(t, t_m)dz$, and by Itoo‘s lemma:
- $\mathrm{dln}[P(t, t_m)] = (\dots)dt + v_m dz$, hence
- $\mathrm{dln}[P(t, t_m)/P(t, t_{k+1})] = (\dots)dt + (v_m - v_{k+1})dz$
- And so the volatility of $w = P(t, t_m)/P(t, t_{k+1})$ is $v_m - v_{k+1}$
- As in one-factor model $\rho = 1$, it holds that with respect to CD account:

$$dF_k(t) = \zeta_k(t)\left(v_{m(t)}(t) - v_{k+1}(t)\right)F_k(t)dt + \zeta_k(t)F_k(t)dz$$

The Libor Market Model

- Since $$\ln P(t,t_k) - \ln P(t,t_{k+1}) = \ln\left(1+\delta_k F_k(t)\right)$$
- using Ito's lemma we obtain

$$v_k(t) - v_{k+1}(t) = \frac{\delta_k \zeta_k(t) F_k(t)}{1+\delta_k F_k(t)}$$

- And so by induction

$$v_m(t) - v_{k+1}(t) = \sum_{i=m}^{k}(v_i - v_{i+1}) = \sum_{i=m}^{k} \frac{\delta_i \zeta_i(t) F_i(t)}{1+\delta_i F_i(t)}$$

$$dF_k(t) = \sum_{i=m}^{k} \frac{\delta_i \zeta_i(t)\zeta_k(t) F_i(t)}{1+\delta_i F_i(t)} F_k(t)dt + \zeta_k(t)F_k(t)dz$$

The model is usually simplified assuming that ζ_k are constant between t_i and t_{i+1}
The volatilities can be obtained e.g. from caplet volatilities.
Starting from initial forward rates the futures rates can be Monte Carlo simulated
To value more complex derivatives e.g. ratchet/sticky/flexi caps, swaptions, etc.
The model can be extended to several independent factors.

LMM Implementation

- Volatilities $\zeta_k(t) = \Lambda_{k-m(t)}$ can be obtained from caplet quotations

$$\sigma_2^2 t_2 = \Lambda_1^2 t_1 + \Lambda_0^2 (t_2 - t_1)$$ Etc.

- Monte Carlo simulation of F_k from t_j to t_{j+1}

Ito $$d \ln F_k(t) = \left(\sum_{i=m(t)}^{k} \frac{\delta_i F_i(t) \Lambda_{i-m(t)} \Lambda_{k-m(t)}}{1 + \delta_i F_i(t)} - \frac{\Lambda_{k-m(t)}^2}{2} \right) dt + \Lambda_{k-m(t)} dz$$

Sample $$\varepsilon_j \approx N(0,1),\ j = 0,1,\ldots$$

and set $$F_k(t_{j+1}) = F_k(t_j) e^{\eta},\ k = j+1,\ldots$$

where $$\eta = \left(\sum_{i=j+1}^{k} \frac{\delta_i F_i(t_j) \Lambda_{i-j-1} \Lambda_{k-j-1}}{1 + \delta_i F_i(t_j)} - \frac{\Lambda_{k-j-1}^2}{2} \right) \delta_j + \Lambda_{k-j-1} \varepsilon_j \sqrt{\delta_j}$$

IR models comparison

- Cerny (2011) in his diploma thesis on „Stochastic Interest Rate Modelling“ (p.84) compares different interest models for the valuation of a complex City of Prague swap entered in 2006

- The valuation results at contract start are as follows:

Source: Author

Model	**Mean PV (mil CZK)**	**Std. Dev. (mil CZK)**
Vasicek	-118.5	13.1
Hull-White	-131.8	18.6
Ho-Lee	-108.6	99.5
LMM	-98.3	124.1

- We can see that while all of the models estimated the value as strongly negative, standard deviation predicted by Ho-Lee and LMM model are much greater than for Vasicek and Hull-White

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