FINANCIAL ENGINEERING
KBP FFÚ

Financial Derivatives II
Part 3

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EVROPSKÁ UNIE
Evropské strukturální a investiční fondy
Operační program Výzkum, vývoj a vzdělávání

MINISTERSTVO ŠKOLSTVÍ, MLÁDEŽE A TĚLOVÝCHOVY

Content

- ✓ Introduction – overview of B.-S. option pricing and hedging
- ✓ Market Risk Management
- ✓ Estimating volatilities and correlations
- ✓ Interest Rate Derivatives Pricing- Martingale and measures
- ✓ Standard Market Model

Content

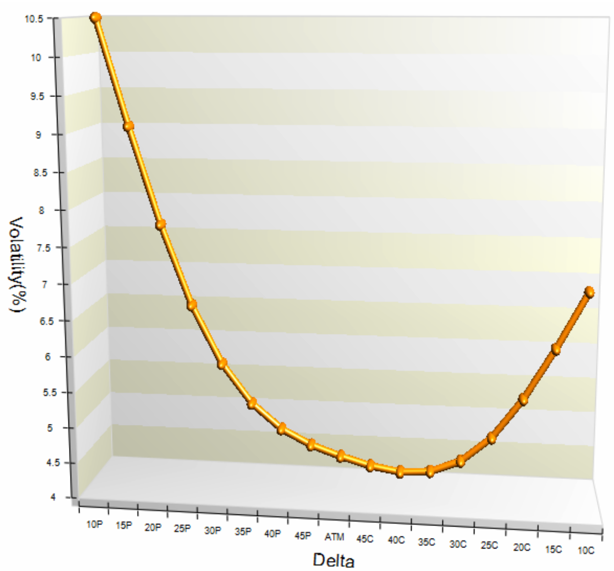
- ✓ Convexity, time, and quanto adjustments
- ✓ Short-rate and advanced interest rate models
- ➢ Volatility smile
- • Exotic options
- • Alternative stochastic models
- • Numerical methods for option pricing
- • Credit derivatives

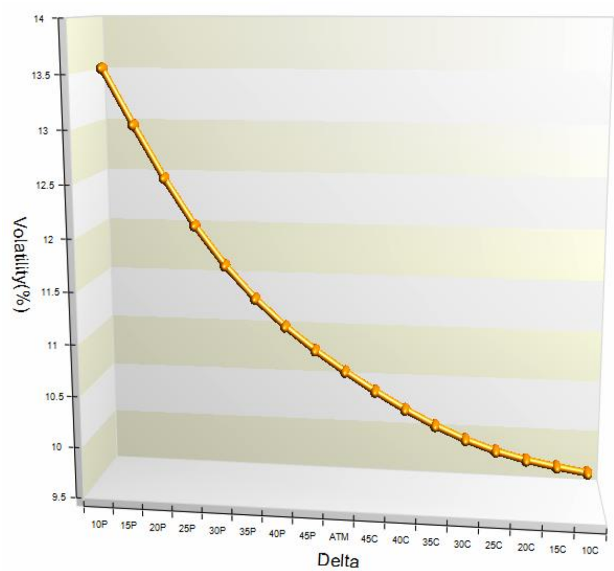
Volatility Smile

- The assumption of the Black-Scholes model that the asset price follows Geometric Brownian Motion with constant volatility leads to biased valuation
- When the B-S model is reversely used to calculate implied volatility from the observed option prices, we can observe a so called **volatility smile** effect

Volatility Smile

- Observed volatility smile for foreign currency options (EUR/CHF, EUR/USD)





It follows from the put-call parity that the smile is identical for calls and puts:

$$c_{\text{mkt}} - p_{\text{mkt}} = c_{\text{BS}}(\sigma) - p_{\text{BS}}(\sigma)$$

Source: MATLAB

Example: calculate the implied volatilies…

Euro FX Option (European) Quotes

Globex

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Quotes | Settlements | Volume | Time & Sales | Contract Specs | Margins | Calendar

Globex Futures | Open Outcry Futures | Globex Options | Open Outcry Options

Auto Refresh is ON

Market data is delayed by at least 10 minutes

Underlying Future	Charts	Last	Change	Prior Settle	High	Low	Volume	Hi / Lo Limit	Updated
Dec 2014		1.2460	-0.0013	1.2473	1.2497	1.2456	41,304	No Limit	03:14:23 CT 26 Nov 2014

Type: European Options

Expiration: Dec 2014

Strike Range: At The Money

CALLS									PUTS								
Updated	Hi / Lo Limit	Volume	High	Low	Prior Settle	Change	Last	Strike Price	Last	Change	Prior Settle	Low	High	Volume	Hi / Lo Limit	Updated
18:14:24 CT 25 Nov 2014	No Limit / 0.00005	0	-	-	0.0242	-	-	12250.0	0.0019 b	0.0000	0.0019	0.0016 a	0.0019 b	0	No Limit / 0.00005	03:10:58 CT 26 Nov 2014
18:14:12 CT 25 Nov 2014	No Limit / 0.00005	0	-	-	0.0201	-	-	12300.0	0.0028 b	0.0000	0.0028	0.0023 a	0.0028 b	0	No Limit / 0.00005	03:10:56 CT 26 Nov 2014
18:13:29 CT 25 Nov 2014	No Limit / 0.00005	0	-	-	0.0162	-	-	12350.0	0.0041 b	+0.0002	0.0039	0.0033 a	0.0041 b	0	No Limit / 0.00005	03:12:05 CT 26 Nov 2014
03:13:30 CT 26 Nov 2014	No Limit / 0.00005	0	0.0138 b	0.0117 a	0.0127	-0.0009	0.0118 b	12400.0	0.0057 b	+0.0003	0.0054	0.0046 a	0.0057 b	0	No Limit / 0.00005	03:12:00 CT 26 Nov 2014
03:13:30 CT 26 Nov 2014	No Limit / 0.00005	0	0.0106 b	0.0087 a	0.0096	-0.0008	0.0088 b	12450.0	0.0077 b	+0.0004	0.0073	0.0063 a	0.0077 b	0	No Limit / 0.00005	03:10:58 CT 26 Nov 2014
03:13:30 CT 26 Nov 2014	No Limit / 0.00005	0	0.0077 b	0.0062 a	0.0070	-0.0007	0.0063 b	12500.0	0.0101 a	+0.0004	0.0097	0.0085 a	0.0103 b	0	No Limit / 0.00005	03:13:30 CT 26 Nov 2014
03:10:58 CT 26 Nov 2014	No Limit / 0.00005	0	0.0054 b	0.0043 a	0.0049	-0.0006	0.0043 a	12550.0	-	-	0.0126	-	-	0	No Limit / 0.00005	01:10:00 CT 26 Nov 2014
03:10:19 CT 26 Nov 2014	No Limit / 0.00005	0	0.0036 b	0.0029 a	0.0033	-0.0004	0.0029 a	12600.0	-	-	0.0160	-	-	0	No Limit / 0.00005	01:10:00 CT 26 Nov 2014
03:10:04 CT 26 Nov	No Limit / 0.00005	0	0.0022 b	0.0019 a	0.0021	-0.0002	0.0019 a	12650.0	-	-	0.0198	-	-	0	No Limit / 0.00005	01:10:00 CT 26 Nov

Source: Globex

Implied (Empirical) versus Lognormal Distribution

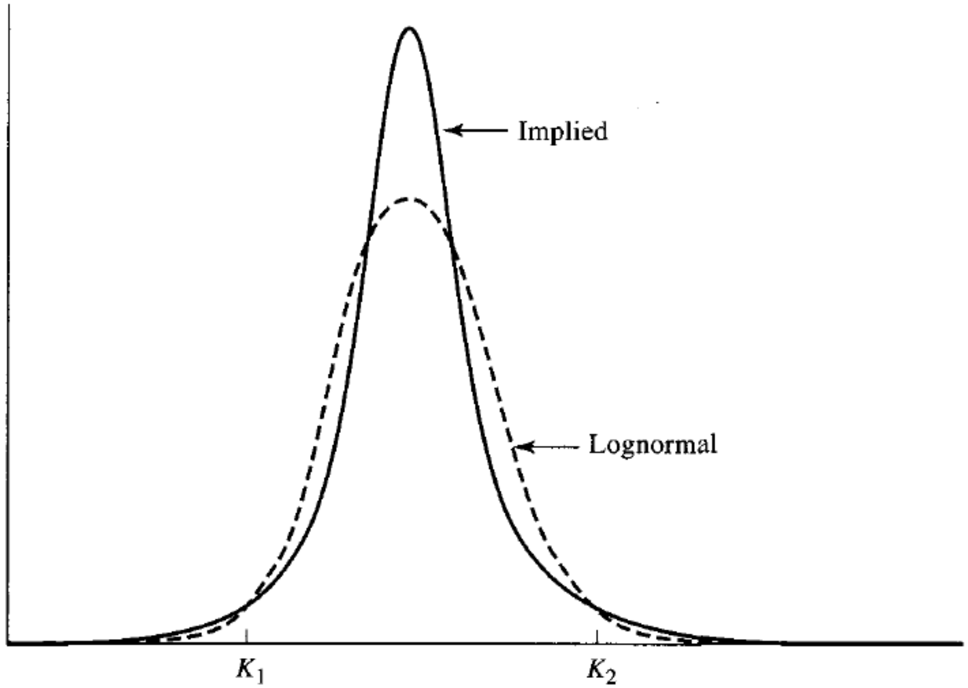


Figure 15.2 Implied distribution and lognormal distribution for foreign currency options

Reasons: volatile volatility, existence of jumps

Source: John Hull, Options, Futures, and Other Derivatives, 5th edition

Implied probability distribution

$$c(K) = e^{-rT} E_T\left[(S_T - K)^+\right] = e^{-rT}\int_K^{\infty}(S_T - K)g(S_T)dS_T$$

Differentiate twice w.r.t. K

$$\frac{\partial c}{\partial K} = -e^{-rT}(K-K)g(K) - e^{-rT}\int_K^{\infty} g(S_T)dS_T = -e^{-rT}\int_K^{\infty} g(S_T)dS_T$$

$$\frac{\partial^2 c}{\partial K^2} = e^{-rT} g(K)$$

…and express the density function using the 2nd order derivative

$$g(K) = e^{rT}\frac{\partial^2 c}{\partial K^2}$$

Equity Options – Volatility Skew

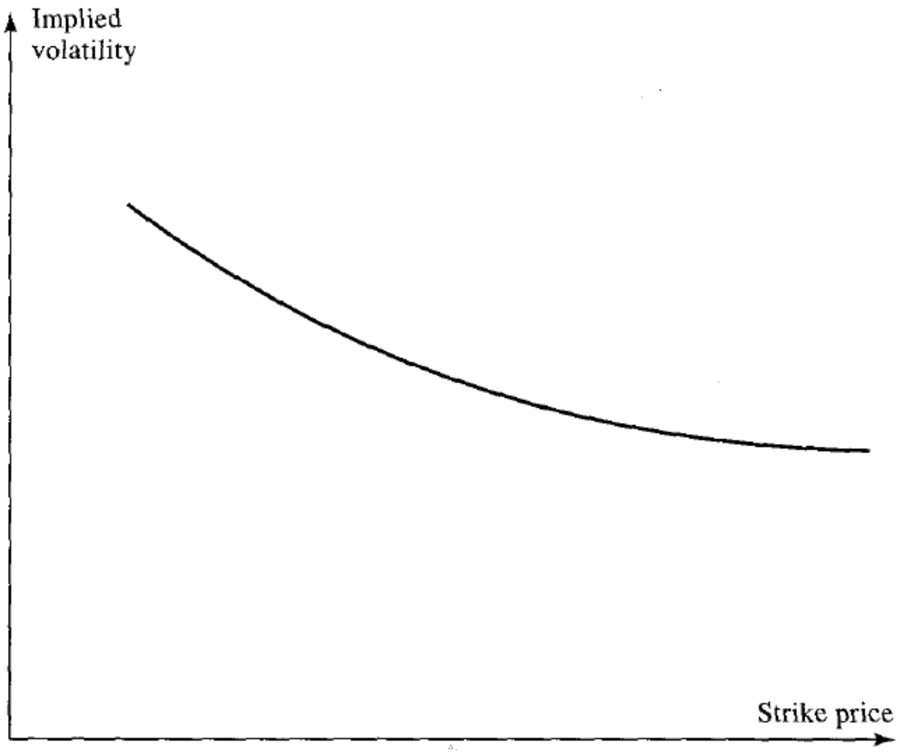


Figure 15.3 Volatility smile for equities

Source: John Hull, Options, Futures, and Other Derivatives, 5th edition

Equity Options - Implied Volatility

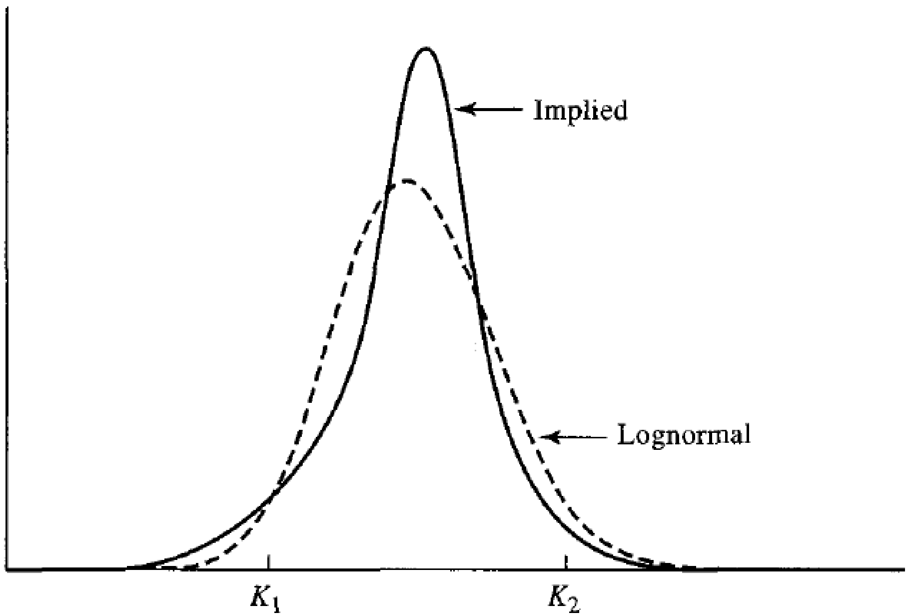
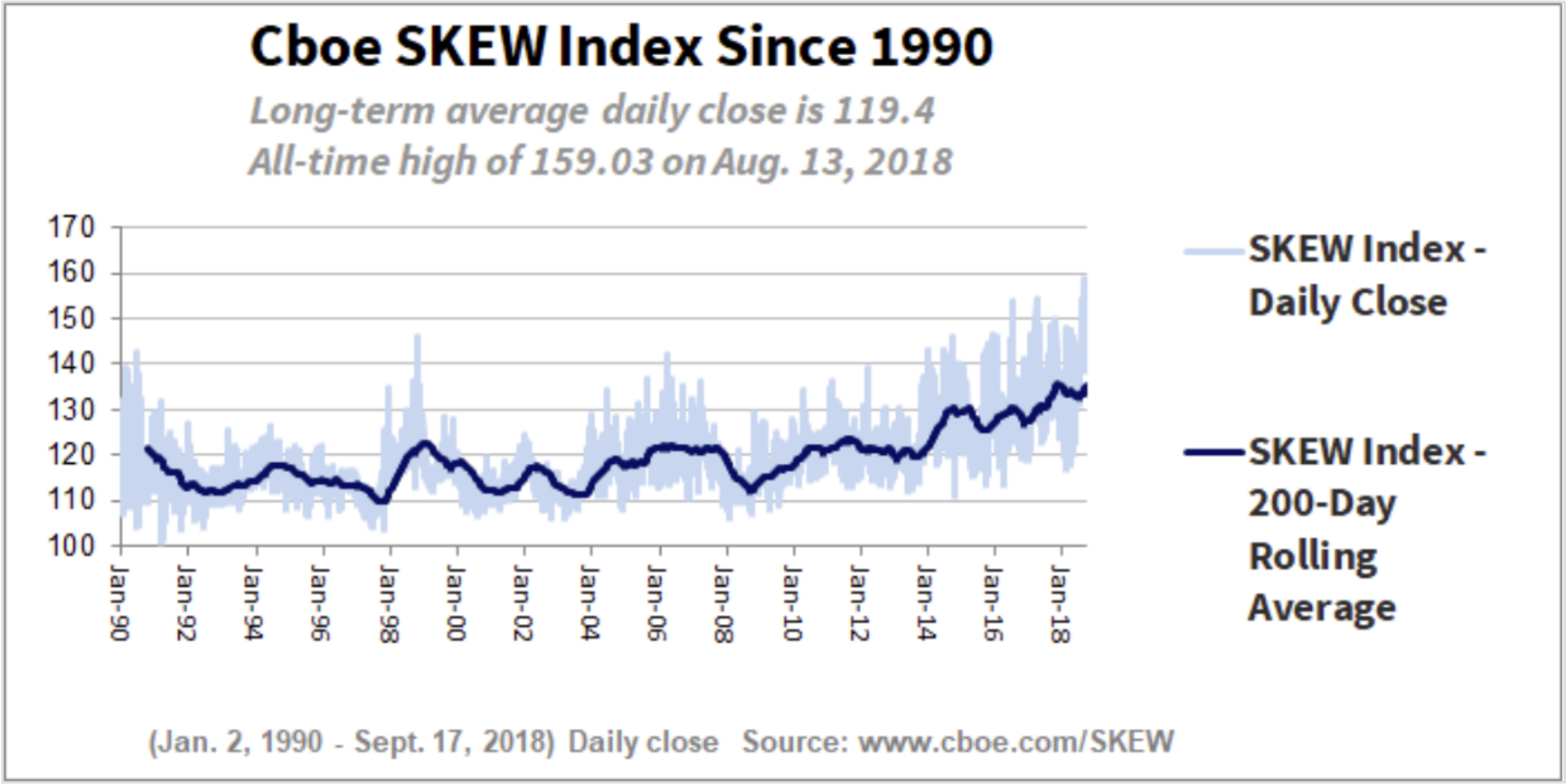


Figure 15.4 Implied distribution and lognormal distribution for equity options

Possible explanation: jumps down more probable than up, negative correlation between volatility and returns, decline of equity implies higher leverage of the company and higher price volatility

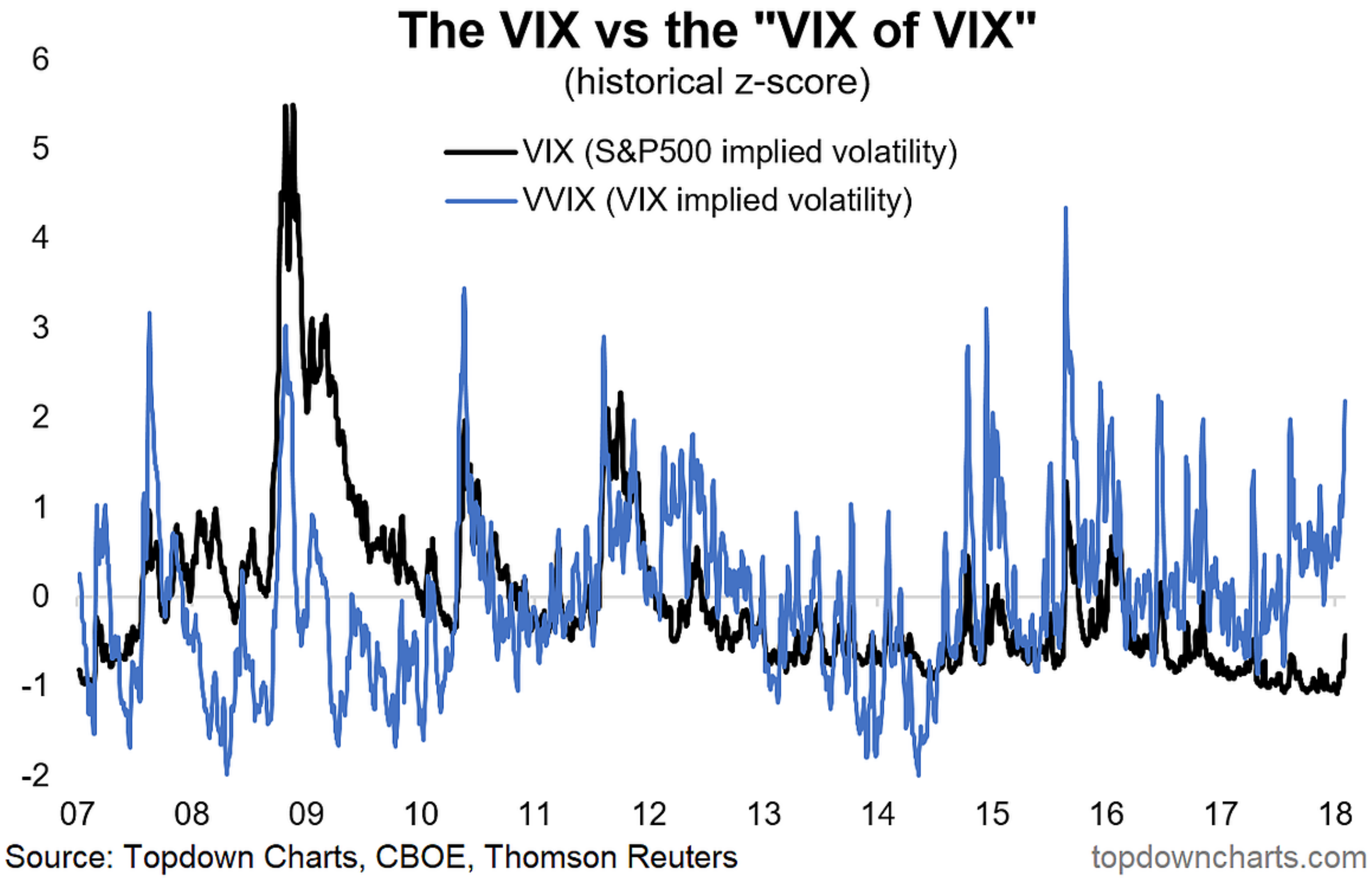
Source: John Hull, Options, Futures, and Other Derivatives, 5th edition

CBOE Skew index



- Skewness index derived by CBOE from the prices of S&P500 out-of-the-money options
- We can see that the skewness of S&P500 returns is increasing since 2008

CBOE VVIX index



VVIX index measures the implied volatility of VIX options (i.e. volatility of volatility, which is related to S&P500 kurtosis)

Volatility Term Structure

- Volatility is quoted as a function of maturity

EURCZKVOL EURCZK FX VOL LINKED DISPLAYS MONEY

	EURCZK				DEALING
SW	27.0	29.0	BROKER	GFX	17:15
1M	25.0	29.0	BROKER	GFX	17:15
2M	23.0	27.0	BROKER	GFX	17:15
3M	20.5	23.5	BROKER	GFX	17:15
6M	18.75	19.25	BROKER	GFX	17:15
9M	17.25	19.0	BROKER	GFX	17:15
1Y	16.25	17.5	BROKER	GFX	17:15

Source: Author

- Caused by the time-varying volatility

Volatility Surface

- Together with the smile there is a volatility surface

Table 15.2 Volatility surface

	Strike price				
	0.90	*0.95*	*1.00*	*1.05*	*1.10*
1 month	14.2	13.0	12.0	13.1	14.5
3 month	14.0	13.0	12.0	13.1	14.2
6 month	14.1	13.3	12.5	13.4	14.3
1 year	14.7	14.0	13.5	14.0	14.8
2 year	15.0	14.4	14.0	14.5	15.1
5 year	14.8	14.6	14.4	14.7	15.0

- The volatility smile complicates the calculation of Greeks – volatility is sensitive to the spot price

Source: John Hull, Options, Futures, and Other Derivatives, 5th edition

Volatility Surface EURUSD

EURVOLSURF EUR VOLSURFACE
Tue 14 May 2013 10:10 GMT Standard Time
Logical Displays <0#EURVOL=R> <0#EURVOLSURF>

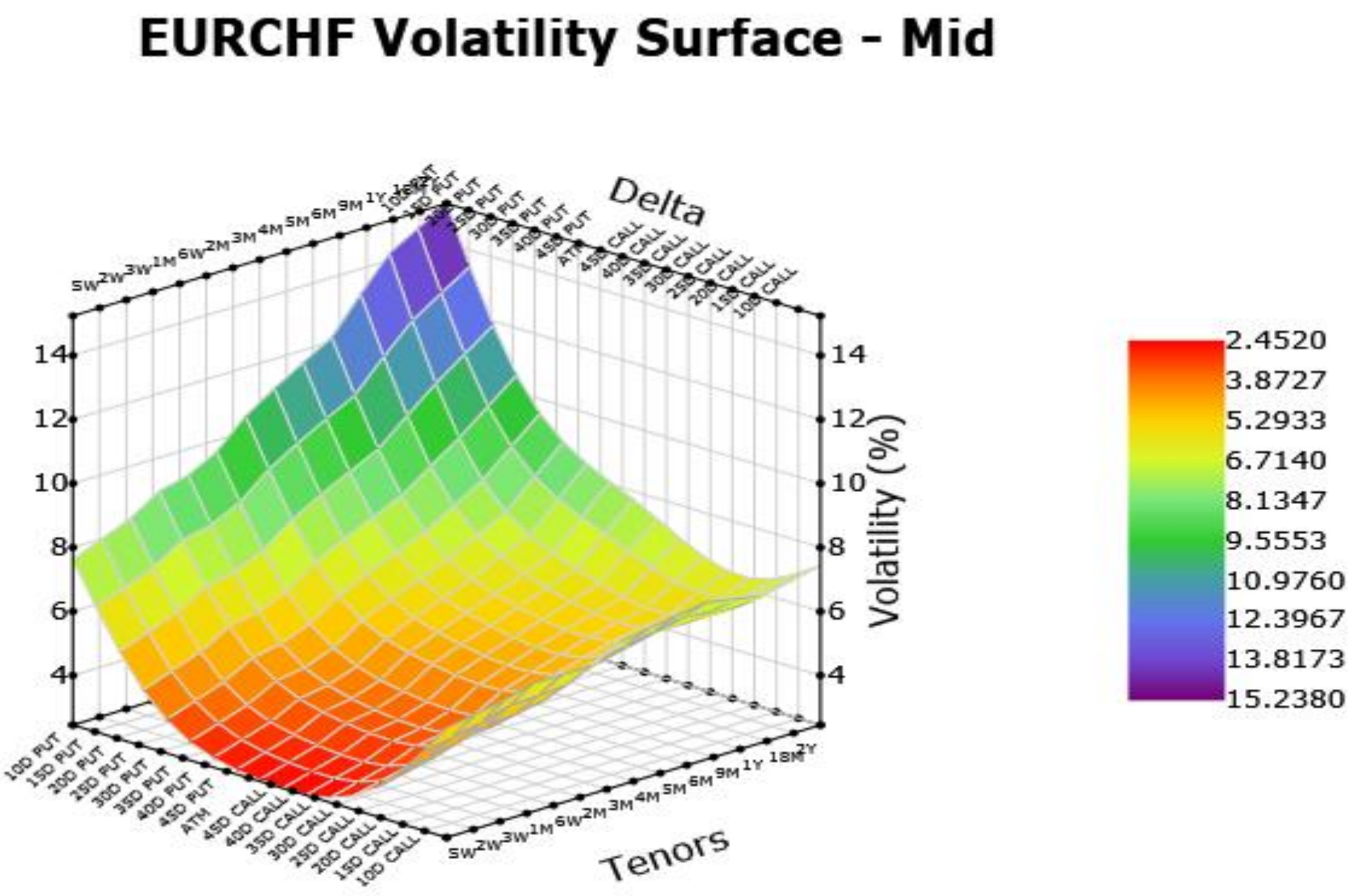
--------------MID--------------

	10DPut	15DPut	20DPut	25DPut	30DPut	35DPut	40DPut	50DPut	ATM
SW	8.676	8.511	8.354	8.213	8.091	7.989	7.903	7.829	7.763
2W	9.064	8.867	8.678	8.509	8.364	8.242	8.139	8.049	7.969
3W	9.189	8.981	8.784	8.605	8.453	8.325	8.216	8.121	8.035
1M	9.263	9.049	8.846	8.663	8.506	8.374	8.261	8.163	8.075
6W	9.423	9.181	8.950	8.743	8.567	8.419	8.294	8.186	8.090
2M	9.537	9.275	9.025	8.800	8.610	8.452	8.318	8.203	8.100
3M	9.850	9.533	9.232	8.963	8.738	8.552	8.398	8.267	8.150
4M	10.173	9.812	9.470	9.166	8.914	8.709	8.540	8.397	8.271
5M	10.382	9.993	9.625	9.298	9.029	8.811	8.632	8.482	8.350
6M	10.513	10.106	9.722	9.382	9.101	8.875	8.691	8.536	8.400
9M	11.000	10.541	10.107	9.725	9.412	9.162	8.960	8.793	8.650
1Y	11.262	10.756	10.279	9.862	9.526	9.261	9.052	8.881	8.737
18M	11.715	11.199	10.713	10.290	9.951	9.685	9.474	9.302	9.168
2Y	11.934	11.412	10.922	10.496	10.155	9.888	9.678	9.505	9.375
3Y	12.360	11.841	11.356	10.937	10.602	10.340	10.132	9.961	9.850
4Y	12.652	12.128	11.639	11.220	10.887	10.625	10.418	01246	10.165
5Y	12.825	12.297	11.806	11.387	11.054	10.793	10.586	01414	10.350
6Y	13.029	12.521	12.044	11.625	11.279	10.995	10.765	01576	10.555
7Y	13.174	12.678	12.210	11.792	11.436	11.137	10.891	01690	10.700
8Y	13.200	12.688	12.209	11.786	11.430	11.133	10.892	01698	10.763
9Y	13.220	12.696	12.208	11.782	11.425	11.130	10.893	01703	10.811
10Y	13.237	12.702	12.207	11.778	11.421	11.128	10.894	01708	10.850

	45DCall	40DCall	35DCall	30DCall	25DCall	20DCall	15DCall	10DCall
SW	7.703	7.650	7.603	7.566	7.537	7.519	7.508	7.501
2W	7.894	7.825	7.765	7.715	7.676	7.651	7.635	7.625
3W	7.956	7.882	7.817	7.763	7.722	7.695	7.677	7.664
1M	7.992	7.916	7.849	7.793	7.750	7.720	7.701	7.688
6W	7.999	7.916	7.842	7.781	7.735	7.705	7.686	7.673
2M	8.004	7.915	7.838	7.773	7.725	7.694	7.675	7.663
3M	8.041	7.942	7.854	7.780	7.725	7.689	7.666	7.650
4M	8.153	8.044	7.948	7.870	7.813	7.779	7.762	7.753
5M	8.226	8.111	8.010	7.928	7.870	7.838	7.824	7.820
6M	8.272	8.153	8.049	7.965	7.906	7.875	7.864	7.863
9M	8.517	8.391	8.283	8.197	8.137	8.107	8.098	8.100
1Y	8.601	8.470	8.357	8.269	8.212	8.191	8.195	8.212
18M	9.040	8.904	8.785	8.689	8.621	8.586	8.575	8.576
2Y	9.250	9.112	8.990	8.890	8.818	8.776	8.758	8.751
3Y	9.745	9.609	9.490	9.392	9.321	9.282	9.266	9.263
4Y	10.088	9.946	9.819	9.711	9.624	9.562	9.520	9.487
5Y	10.288	10.144	10.013	9.898	9.802	9.727	9.669	9.620
6Y	10.537	10.374	10.235	10.117	10.013	9.920	9.830	9.742
7Y	10.711	10.535	10.391	10.270	10.161	10.054	9.943	9.829
8Y	10.844	10.648	10.490	10.358	10.244	10.134	01020	9.901
9Y	10.947	10.736	10.566	10.427	10.308	10.195	01078	9.957
10Y	11.029	10.806	10.627	10.482	10.359	10.244	01125	10.001

Source: Author

Volatility Surface Example



Source: Thomson Reuters

Single Large Jump

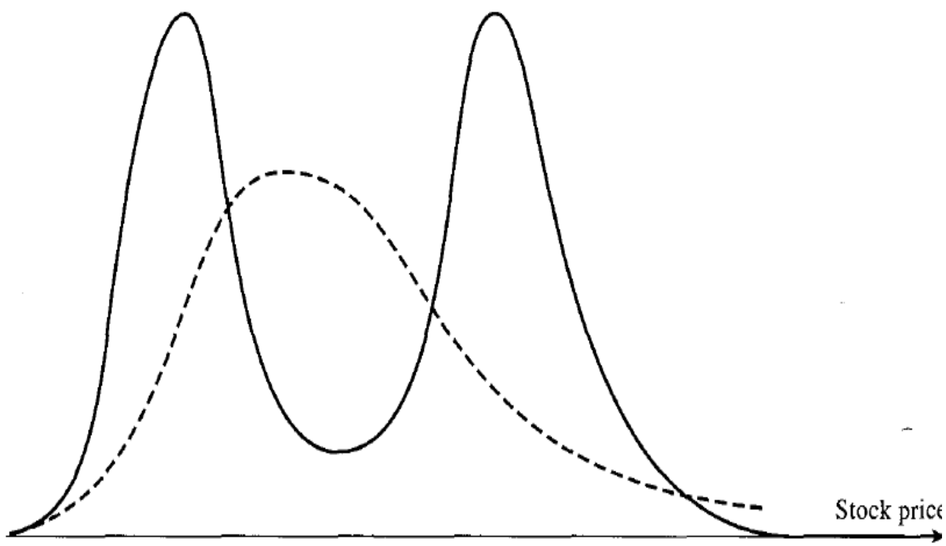
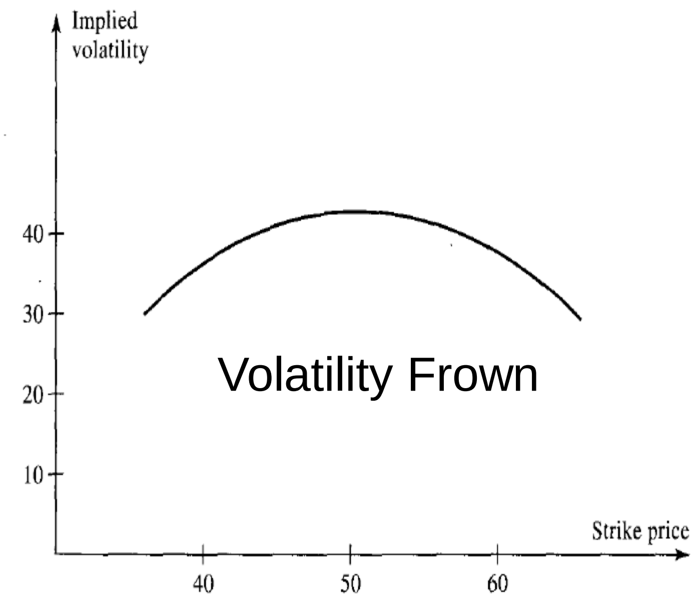


Figure 15.5 Effect of a single large jump: the solid line is the true distribution the dashed line is the lognormal distribution



Volatility Frown (i.e. concave volatility smile) is also commonly observed in the case of mean-reverting assets such as interest rates or VIX

Source: John Hull, Options, Futures, and Other Derivatives, 5th edition, Author

Model Free Volatility

- Valid for a wide range of price processes
- Expected integrated variance, i.e. $E\left[\int_0^T \sigma_t^2 dt\right]$
- According to Neuberger and Britten-Jones (2000) can be derived from the continuum of option prices $C(T,K)$

$$E_0^F\left[\int_0^T \sigma_t^2 dt\right] = E_0^F\left[\int_0^T \left(\frac{dF_t}{F_t}\right)^2 dt\right] = 2\int_0^\infty \frac{C^F(T,K) - \max(0, F_0 - K)}{K^2} dK$$

- The result holds also in presence of jumps (Jiang and Tian, 2005)
- Since 2003 it is used for VIX calculation

Volatility and variance swaps

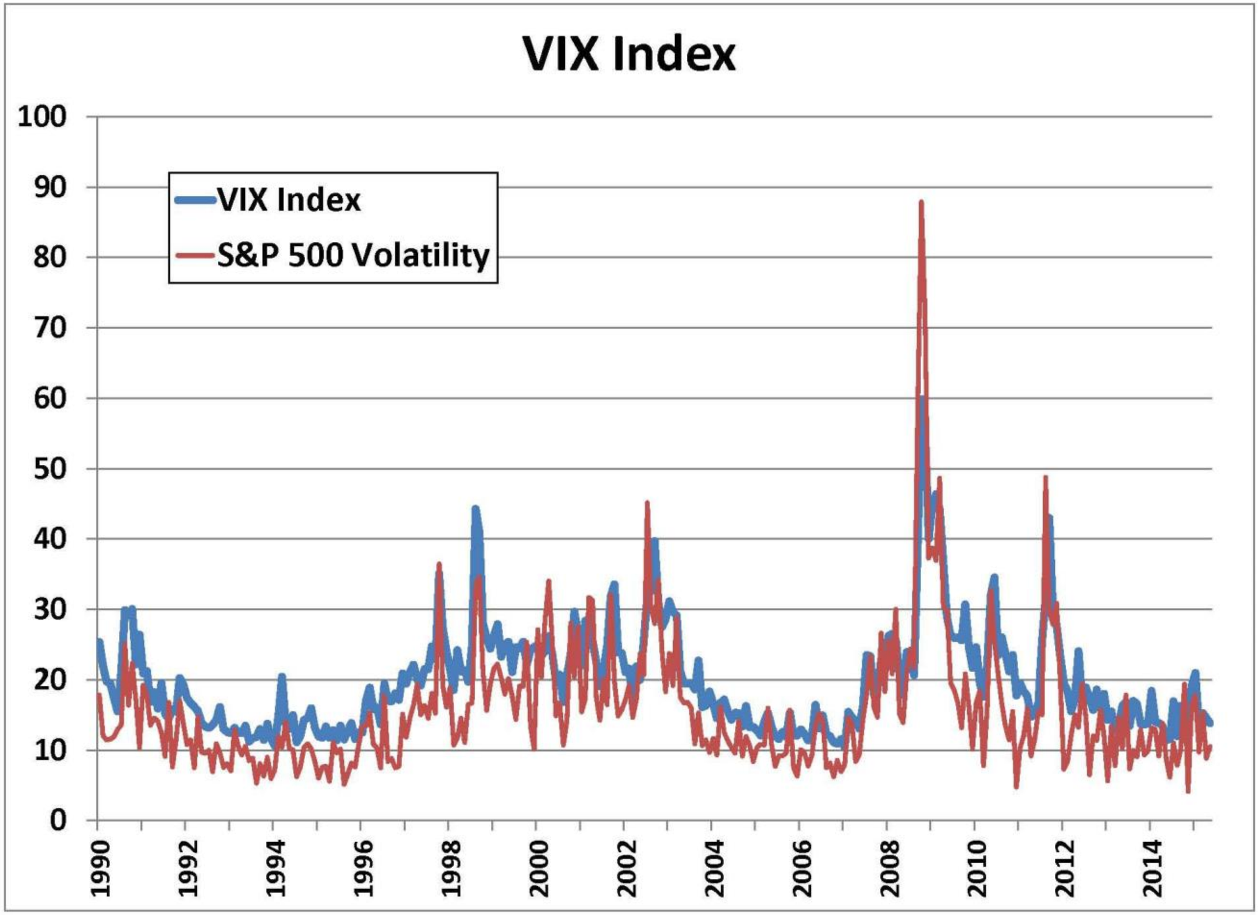
- **Volatility swap** – payoff depends on the difference between strike volatility and realized volatility until the maturity of the swap
- **Variance swap** – payoff depends on the difference between strike and realized variance
- Represent instruments used to directly enter long/short positions in volatility (alternative to straddles, strangles, VIX futures, etc.)
- While for volatility swap there is no closed-form valuation formula, the ekvilibrium rate of a variance seap is known and is equal to the model-free variance:

$$E_0^F\left[\int_0^T \sigma_t^2 dt\right] = E_0^F\left[\int_0^T \left(\frac{dF_t}{F_t}\right)^2 dt\right] = 2\int_0^\infty \frac{C^F(T,K) - \max(0, F_0 - K)}{K^2} dK$$

Variance risk premium

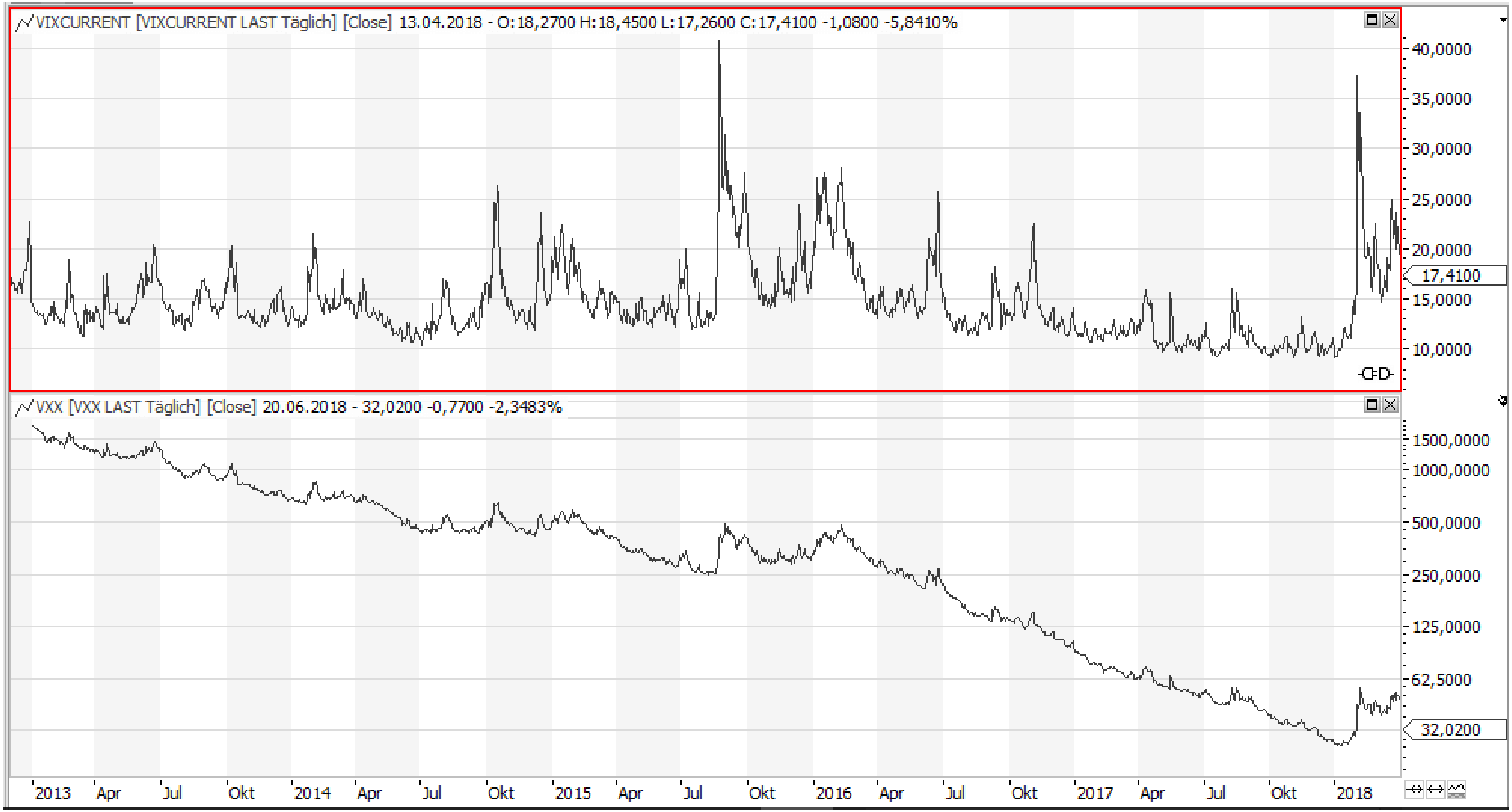
- It turns out that implied volatility (and model-free and variance swap rates) tend to systematically overestimate the future realized volatility
- The effect is known as **variance risk premium**
- Explanations:
  - Volatility has strongly negative correlations with market returns (negative beta), investors with short position in volatility (i.e. short option positions) demand positive risk premium in order to be in these positions
  - Volatility exhibits asymmetry (skweness) upwards (i.e. potential of increase is larger than of a decrease), leading to a skewness risk premium (for short positions)
  - Jumps in volatility tend to typically occurr upwards

Variance risk premium - VIX



Source: Author

Variance risk premium - VXX



Volatility risk-premium is contained also in VIX futures – The chart shows the profit of VXX ETF which constantly shorts VIX futures

Content

- ✓ Convexity, time, and quanto adjustments
- ✓ Short-rate and advanced interest rate models
- ✓ Volatility smiles
- ➢ Exotic options
- Alternative stochastic models
- Numerical methods for option pricing
- Credit derivatives

Exotic Options

- **Classification (Wilmott):**
- Path dependence (weak, strong – new variable must be introduced into valuation, discrete, continuous)
- Dimensionality (multi factor, strong path dependence)
- Order
- Embedded decisions
- Cash flows (discrete, continuous)
- Time dependence

Exotic Options

- Nonstandard American Options, e.g. *Bermudan Options* – exercise restricted to certain dates – time dependence and embedded decision example (valuation using binomial trees)
- *Compound options* – options on options (European compound options can be valued analytically)
- *Chooser Options* – at certain time the holder specifies whether it is a put or call (European style can be valued analytically using the put-call parity)

$$\max(c, p) = \max(c,\ c + Ke^{-r(T_2-T_1)} - S_1 e^{-q(T_2-T_1)})$$

$$= c + e^{-q(T_2-T_1)} \max(0,\ Ke^{-(r-q)(T_2-T_1)} - S_1)$$

Binary options

- Payoff 0 or a fixed amount Q (or the asset)
- European style can be valued analytically using the risk-neutral valuation principle where $P[S_T > K] = N(d_2)$

$$c_{bin,cash} = Qe^{-rT}N(d_2) \qquad p_{bin,cash} = Qe^{-rT}N(-d_2)$$

$$c_{bin,asset} = S_0e^{-qT}N(d_1) \qquad p_{bin,asset} = S_0e^{-qT}N(-d_1)$$

- Normal European call = $+1$ asset-or-nothing call -1 strike price-or-nothing call

$$c = S_0e^{-qT}N(d_1) - Ke^{-rT}N(d_2)$$

Barrier Options

- Payoff depends also on reaching certain barrier during a time period
- *Knock-out (put/call)…* no pay-off if the barrier is reached
- *Knock-in (put/call) …* pay-off only if the barrier is reached
- Down/up-and-in, down/up-and-out (put/call)
- Can be valued analytically (European type) in the context of geometric Brownian motion
- An adjustment necessary if the barrier crossing is observed only in discrete times
- Hedging difficult due to discontinuities

Barrier Options - Types

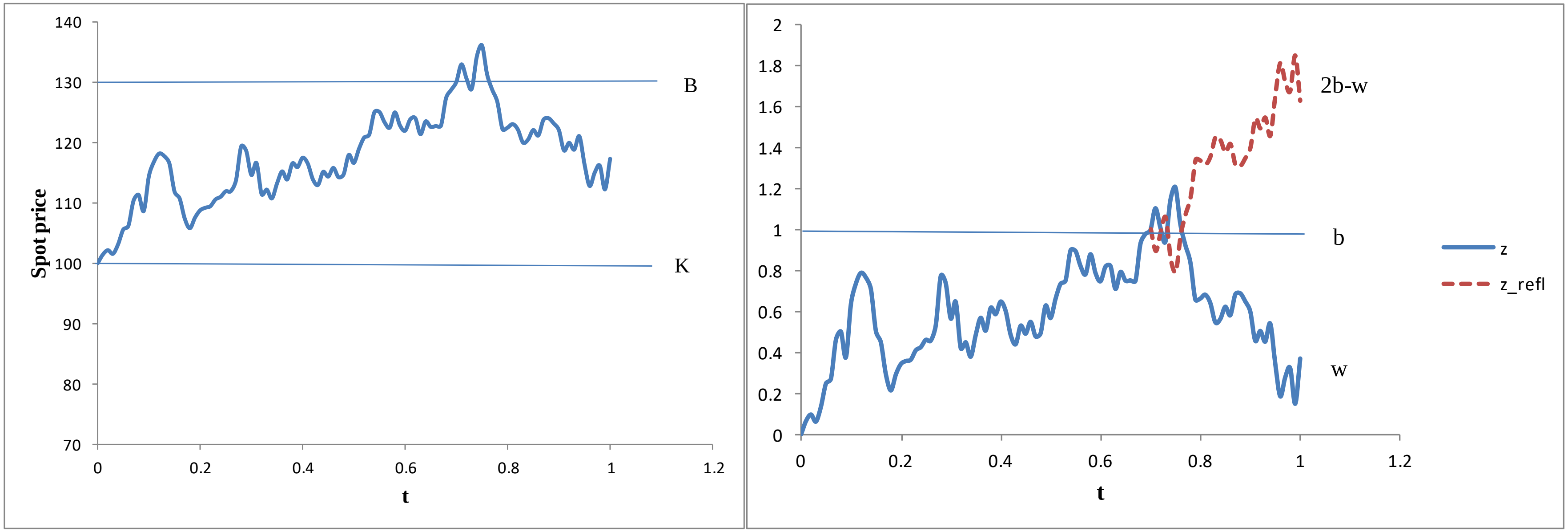
Option	Type	Barrier Location
Call	Up-and-Out	Above Spot
	Up-and-In	Above Spot
	Down-and-Out	Below Spot
	Down-and-In	Below Spot
Put	Up-and-Out	Above Spot
	Up-and-In	Above Spot
	Down-and-Out	Below Spot
	Down-and-In	Below Spot

Source: Author

- As for each type of option we can be in long vs. short position, there are altogether 16 possible positions
- Barrier options can also differ based on whether the breach of barrier is observed at any time until maturity or only at maturity (less common)

Reflection principle

- The key idea to value barrier options assuming the geometric Brownian motion



Source: Author

Valuation Formula – up and out European call (BS model)

$$E\left[(S_T-K)^+ I(M_T<B)\right]=\int_K^B (S-K)\cdot\Pr\left[z(T)\in[w,w+dw)\,\&\,m(T)<b\right]$$

$$\begin{aligned}
c_{uo}(S_0,0) &= S_0\left(N(\delta_+(T,S_0/K)-N(\delta_+(T,S_0/B))\right)- \\
&-e^{-rT}K\left(N(\delta_-(T,S_0/K)-N(\delta_-(T,S_0/B))\right)- \\
&-B(S_0/B)^{-2r/\sigma^2}\left(N(\delta_+(T,B^2/(KS_0))-N(\delta_+(T,S_0/B))\right)+ \\
&+e^{-rT}K(S_0/B)^{-2r/\sigma^2+1}\left(N(\delta_-(T,B^2/(KS_0))-N(\delta_-(T,S_0/B))\right),
\end{aligned}$$

$$\delta_\pm(\tau,s)=\frac{1}{\sigma\sqrt{\tau}}\left[\ln s+\left(r\pm\frac{\sigma^2}{2}\right)\tau\right]$$

Asian Options

$$S_{ave}(t,T) = \frac{1}{T}\int_t^T S(s)ds$$

$$\tilde{S}_{ave}(t,T) = \frac{1}{m}\sum_{j=1}^{m} S(t_j)$$

- Payoff depends on the average price
- Asian price vs. Asian strike options:
  - Asian price call *payoff* = max(S_{ave}-K,0)
  - Asian price put *payoff* = max(K-S_{ave},0)
  - Asian strike call *payoff* = max(S_T -S_{ave},0)
  - Asian strike put *payoff* = max(S_{ave}-S_T,0)
- Useful to hedge an average cost (e.g. exchange rate, comodity prices) during some time period
- Exact analytic formula not available if the average is arithmetic (exists if geometric)
- If the average is geometric, binomial trees can be used for valuation, otherwise – Monte-Carlo simulations

Asian Options – Monte Carlo

- Asian options with arithmetic means usually need to be valued with Monte Carlo simulations
- To value a fixed-strike Asian call, we can:

1. Simulate $i = 1, \dots, N$ evolutions of the stock price for the period $t = 1, \dots, T$, using the equation:

$$S_t^{(i)} = S_{t-1}^{(i)} \exp\left(\mathrm{r} - \frac{\sigma^2}{2} + \sigma \varepsilon_t^{(i)}\right)$$

   starting with the initial stock price S_0, T is the maturity, and $\varepsilon_t^{(i)} \sim N(0,1)$ are simulated values from standard normal distribution

2. For each simulation i compute the $S_{ave,1:T}^{(i)}$ and payoff $c_T^{(i)}$:

$$S_{ave,1:T}^{(i)} = \frac{1}{T} \sum_{t=1}^{T} S_t^{(i)} \qquad c_T^{(i)} = \max\left(S_{ave,1:T}^{(i)} - K, 0\right)$$

3. The estimated value of the option is then equal to:

$$c_0 = e^{-rT} \frac{1}{N} \sum_{i=1}^{N} c_T^{(i)}$$

Lookback Options

- Payoff depends on the maximum or minimum reached during the option life:
  - Floating-Strike Lookback call *payoff* = $\max(S_T - S_{min}, 0)$
  - Floating-Strike Lookback put *payoff* = $\max(S_{max} - S_T, 0)$
  - Fixed-Strike Lookback call *payoff* = $\max(S_{max} - K, 0)$
  - Fixed-Strike Lookback put *payoff* = $\max(K - S_{min}, 0)$
- European type lookbacks can be valued analytically
- The pricing formula involves the modelling of the final stock price as well as the maximum stock price until maturity (derivation of the formula is similar as for barrier options)

Shout Options

- *Shout options* – shout … 1st realization any time during the life, maturity the 2nd, payoff = the maximum (valuation using binomial trees) $\max(S_T - K, S_\tau - K, 0)$

Options Involving Two or More Assets

- Value depends on multiple assets:
  - **One-Asset-For-Another option**
  - **Basket options** – Payoff depends on the average value of a basket of assets
  - **Rainbow options** – Payoff depends on the performance of the best or the worst asset in a basket
  - **Mountain range options** – Complicated payoff, usually depends on the performance of *k* best or worst performing assets in a basket:
    - **Himalayan option** – Payoff based on average of *k* best assets
    - **Everest option** – Payoff based on average of *k* worst assets
    - **Atlas option** – *k* best and *k* worst assets are removed from basket
    - **Annapurna option** – Basket option with a Knock-Out if *k* assets in the basket fall below a certain barrier
- Correlation plays a crucial role in valuation
- Valuation usually with Monte-Carlo simulations

Option to Exchange one asset for another

- One asset for another option *payoff* = max(V_T-U_T,0)
- Valuation based on the change of numeraire technique

$$V_0 e^{-q_V T} N(d_1) - U_0 e^{-q_U T} N(d_2)$$

$$d_1 = \frac{\ln(V_0/U_0) + (q_U - q_V + \hat{\sigma}^2/2)T}{\hat{\sigma}\sqrt{T}}$$

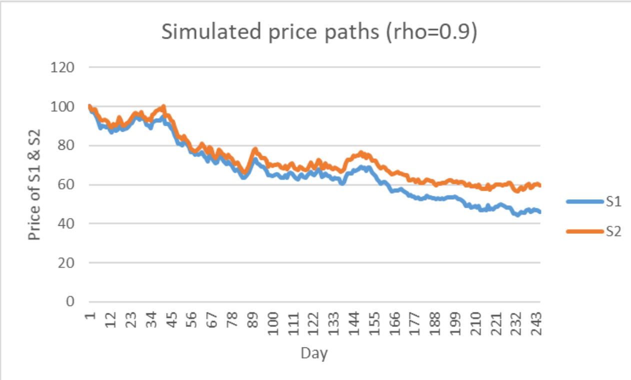
$$d_2 = d_1 - \hat{\sigma}\sqrt{T}$$

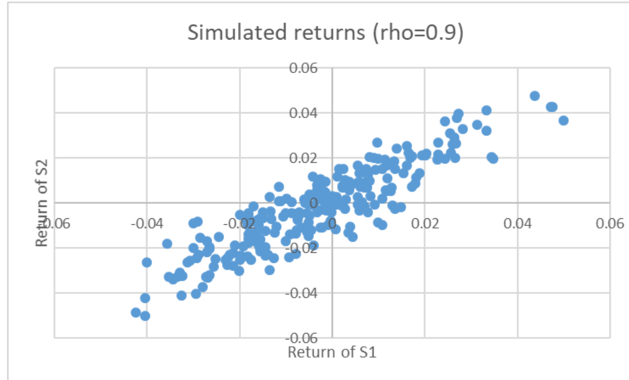
$$\hat{\sigma} = \sqrt{\sigma_U^2 + \sigma_V^2 - 2\rho\sigma_U\sigma_V}$$

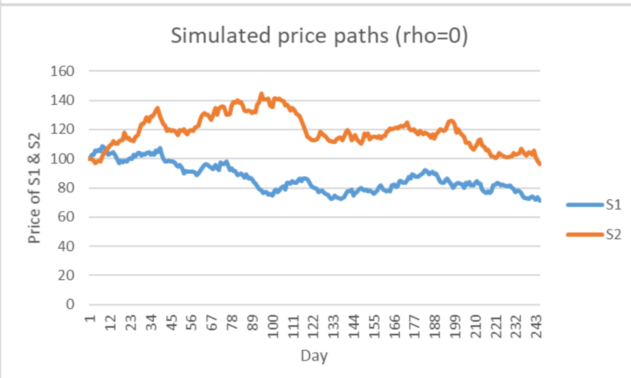
Monte-Carlo valuation of 2-asset options

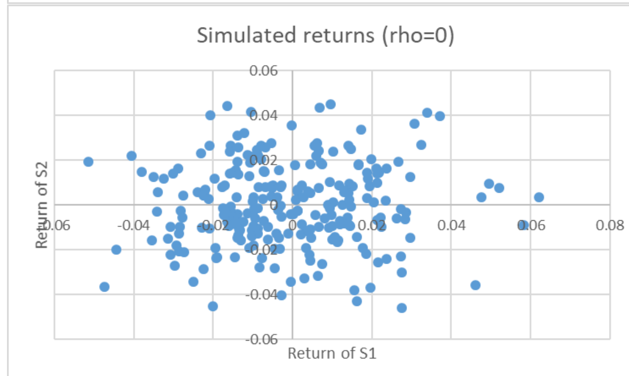
- Correlated Geometric Brownian Motions for 2 assets:
- $dS_{1,t} = \mu_1 S_{1,t} dt + \sigma_1 S_{1,t} dW_{1,t}$
- $dS_{2,t} = \mu_2 S_{2,t} dt + \sigma_2 S_{2,t} dW_{2,t}$
- Where it holds that $\mathrm{E}(dW_{1,t} dW_{2,t}) = \rho dt$
- In order to simulate $dW_{1,t}$ and $dW_{2,t}$, we use:
- $dW_{1,t} \sim N(0, \sqrt{dt})$
- $dW_{2,t} \sim \rho dW_{1,t} + \sqrt{1-\rho^2} dZ_t$
- Where Z_t is a Wiener process uncorrelated with $W_{1,t}$, so:
- $dZ_t \sim N(0, \sqrt{dt})$
- We set $\mu_1 = \mu_2 = r$, simulate the asset paths, value the option at T, and discount with the risk-free rate r

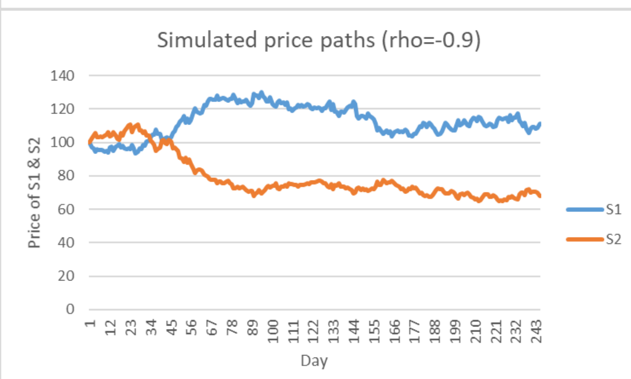
Example - Simulation of correlated returns

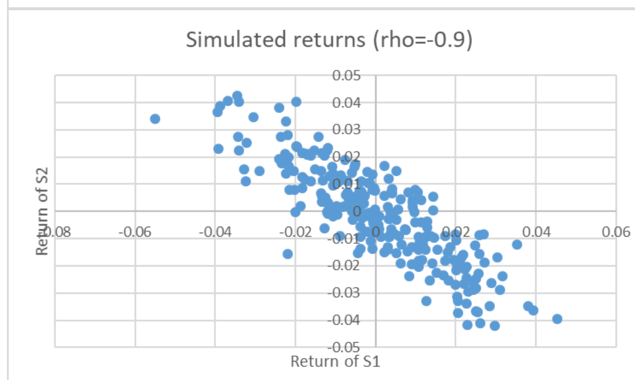












We use parameters:

$$\mu_1 = \mu_2 = r = 2\% * (1/252),$$
$$\sigma_1 = \sigma_2 = 30\% * (1/\sqrt{252})$$
$$S_{1,0} = S_{2,0} = 100$$

And for each period t do:

$$S_{1,t} = S_{1,t-1}\exp\left[\mu_1 - \frac{1}{2}\sigma_1^2 + \sigma_1 W_{1,t}\right]$$

$$S_{1,t} = S_{2,t-1}\exp\left[\mu_2 - \frac{1}{2}\sigma_2^2 + \sigma_2 W_{2,t}\right]$$

$$W_{1,t} \sim N(0,1)$$

$$W_{2,t} \sim \rho W_{1,t} + \sqrt{1-\rho^2} Z_t$$

$$dZ_t \sim N(0,1)$$

Source: Author

Monte-Carlo valuation of *N*-asset options

- Each asset $i = 1, \ldots, N$ follows a process:
- $dS_{i,t} = \mu_i S_{i,t} dt + \sigma_i S_{i,t} dW_{i,t}$
- Which, according to Itoo‘s Lemma gives:
- $S_{i,T} = S_{i,0} \exp\left[\left(\mu_i - \frac{1}{2}\sigma_i^2\right)T + \sigma_i W_{i,T}\right]$
- For correlated assets it holds that $E\left[W_{i,T} W_{j,t}\right] = \Omega_{i,j} T$
- Where $\Omega_{i,j}$ is the correlation between i and j
- How do we generate samples from $W_{i=1,\ldots,N,T}$?
- Suppose X is a vector of independent $N(0,1)$ variables, and $Y = LX$
- Covariance (correlation) matrix of X is given as:
- $E(YY^T) = E(LXX^T L^T) = LE(XX^T)L^T = LL^T,$ Since $E(XX^T) = I$
- To get $E(YY^T) = \Omega$, we need to use Cholesky decomposition to find L
- $LL^T = \Omega$
- Values from the correlated vector $W_{i=1,\ldots,N,T}$ can then be simulated by drawing N independent $N(0,1)$ variables and transforming them with $Y = LX$
- The correlation matrix Ω is typically estimated from historical data

Warrants

- Call options issued by firms, which give the holder the right to purchase shares of the firm at a fixed price.
- Main difference between warrant and call option is that the firm <u>issues new shares if the warrant is exercised</u>
- Exercise of the warrant will thus dilute the firms equity
- The effective payoff of the warrant at maturity T is:

$$payoff = \max\left(\frac{E_T + M * X}{N + M} - X, 0\right)$$

- Where E_T is the value of the firms equity, M is the number of issued warrants, X is the exercise price of warrants, and N is the number of shares outsdtanding prior to the exercise of warrants
- The payoff will thus depend on the overal amount of warrants outstanding M

Warrants - Rearrangement

- In order to derive the valuation formula for warrants, it is usefull the rearrange the payoff function:

$$payoff = \max\left(\frac{E_T + MX}{N + M} - X, 0\right)$$

$$payoff = \max\left(\frac{E_T + MX}{N + M} - \frac{N + M}{N + M}X, 0\right)$$

$$payoff = \max\left(\frac{E_T + MX - NX - MX}{N + M}, 0\right)$$

$$payoff = \max\left(\frac{E_T - NX}{N + M}, 0\right)$$

$$payoff = \frac{N}{N + M}\max\left(\frac{E_T}{N} - X, 0\right)$$

Warrants - Valuation

- The warrant payoff formula can be rearranged into:

$$\frac{N}{N+M}\max\left(\frac{E_T}{N}-X,0\right)$$

- A problem is that E_T must include the value of the warrants
- The value of equity E_0 at time 0 is:

$$E_0 = N * S_0 + M * W_0$$

- Where S_0 is the stock price and W_0 the price of the warrant
- i.e. to value the warrent W_0, the underlying in the option pricing model has to be $S_0 + \frac{E_T}{N} W_0$, which includes the unknown W_0
- The option valuation formula $Call(S, K, T, \sigma, R_f)$, thus has to be applied recursivelly, starting with an initial estimate $W_0^{(0)}$
- We then <u>run the following recusion</u> until the result converges:

$$W_0^{(i)} = \frac{N}{N+M} Call\left(S_0 + \frac{E_T}{N} W_0^{(i-1)}, X, T, \sigma, R_f\right)$$

Content

- ✓ Convexity, time, and quanto adjustments
- ✓ Short-rate and advanced interest rate models
- ✓ Volatility smiles
- ✓ Exotic options
- ➢ Alternative stochastic models
- Numerical methods for option pricing
- Credit derivatives

Alternative Stochastic Models

- Empirical observations differ from the lognormal returns assumption – volatility surface
- Need of alternative models in particular for exotic (e.g. barrier) options

❖ Diffusion models – prices change continuously

❖ Mixed jump-diffusion models

❖ Pure jump models

❖ Stochastic Volatility models (without/with jumps)

❖ Variance-Gamma model

Constant Elatisticity of Variance Model

$$dS = (r - q)Sdt + \sigma S^{\alpha} dz$$

- $\alpha = 1$ GBM, $\alpha < 1$ heavy left tail, $\alpha > 1$ heavy right tail
- Analytic formulas exist for European call and put
- Applicable to options on equity or futures (skew), not FX (smile)
- For exotic options parameters are fit to prices of plain vanilla options minimizing the sum of squared differences

Implied Volatility Function (IVF) Models

$$dS = \left(r(t) - q(t)\right)Sdt + \sigma(S,t)Sdz$$

- The ***local volatility*** function $\sigma(S,t)$ is chosen to price all (plain vanilla) European options consistently with the market
- It is also called “the implied tree” as the volatilities can be estimated step by step on nodes of a binomial tree
- Joint distributions of prices at different times can be modeled incorrectly – problem for compound, barrier, and some other exotic options

Mixed Jump Diffussion Model

$$dS = (r - q - \lambda m_J)Sdt + \sigma Sdz + dJ$$

- Probability of a jump in time *dt* is λdt
- The jump process is usually decomposed as $dJ = \left(a_t - 1\right)SdN$ where *dN* is the Poisson counting process and a_t *-1 the jump size*
- Average jump size as a percentage of *S* is m_J
- The Wiener process *dz* and the jump process *dJ* are independent
- If a_t is lognormally distributed then there is a formula (Merton) – an infinite series involving B-S prices, not very nice
- Heavier left and right tails - appropriate for FX options

Merton's Formula

$$f = \sum_{n=0}^{\infty} \frac{e^{-\tilde{\lambda}T}(\tilde{\lambda}T)^n}{n!} f_n$$

- Where f_n is the option value conditional on n jumps with the adjusted volatility and the risk free rate:

$$\sigma_n = \sqrt{\sigma^2 + \frac{ns^2}{T}}, \; r_n = r - \lambda m_J + \frac{n\ln(1+m_J)}{T}$$

- Since the drift is not r the lambda needs to be adjusted (due to change of measure) to $\tilde{\lambda} = \lambda + \lambda m_J$
- $\ln S(T)$ jump distribution: $N(\ln(1+m_J), s^2)$

Stochastic Volatility Models

$$\frac{dS}{S} = (r - q)dt + \sqrt{V}dz$$

e.g. $$dV = a(V_L - V)dt + \xi V^{\alpha} dz_V, \text{ or}$$

$$d\log V = \kappa(\theta - \log V)dt + \sigma_V dz_V$$

- The stochastic variable *V* models the variance
- There is a mean reversion of *V* to *a* long-term mean
- If the Wiener processes dz_S and dz_V are uncorrelated and *V* follows the GBM process then there is a semianalytic formula for European options (Hull, White)
- If α = 0.5 then there is a semianalytic formula (Heston)
- Otherwise a Monte Carlo simulation must be used
- Parameters typically fitted to historical returns and/or to prices of plain vanilla options and then used to price exotic options (e.g. FX)

Hull-White Lemma

- Suppose $dV = \alpha V dt + \xi V dz_V$ with dz and dV independent and

$$\bar{V} = \frac{1}{T}\int_0^T V dt$$

- Then $\ln(S_T / S_0) \sim N(rT - \bar{V}T/2, \bar{V}T)$ is lognormally distributed conditional upon $\bar{V}$

Hull White Formula

- Option value can be expressed as a probability density weighted mean of the option values conditional on $\bar{V}$

$$f(S_t,\ \sigma_t^2,\ t) = \int \left[e^{-r(T-t)} \int f(S_T) g(S_T|\bar{V}) dS_T \right] h(\bar{V}|\sigma_t^2) d\bar{V}.$$

- Assuming that $\bar{V}$ has a known distribution we integrate the BS formula over the distribution
- Hull, White (1987) use the Taylor expansion of BS and known moments of $\bar{V}$ if V follows the GBM

Heston model

- Stochastic volatility model with semi-analytical solution for the option price
- Variance follows the CIR process
- $dS_t = \mu S_t dt + \sqrt{V_t} S_t dW_{S,t}$
- $dV_t = \kappa\,(\theta - V_t)dt + \xi\sqrt{V_t}dW_{V,t}$
- $W_{S,t}$ and $W_{V,t}$ are correlated Wiener processes with correlation ρ
- Similarly to the Black-Scholes PDE, we can derive the Heston PDE (uses Vanna and Vomma):
- $\frac{df}{dt} + \frac{1}{2}VS^2\frac{\partial^2 f}{\partial S^2} + \rho\sigma VS\frac{\partial^2 f}{\partial V \partial S} + \frac{1}{2}\xi^2 V\frac{\partial^2 f}{\partial V^2} + (r-q)S\frac{\partial f}{\partial S} + [\kappa\,(\theta - V)]\;S\frac{\partial f}{\partial V} = rf$

Heston calibration - Example

- The EUR/USD spot price and the volatility term structure for EUR/USD call options was downloaded from Investing.com:

1W		1M		3M		6M		12M	
Strike	Vol	Strike	Vol	Strike	Vol	Strike	Vol	Strike	Vol
1.126	5.88%	1.12	5.57%	1.11	6.11%	1.09	6.95%	1.09	7.49%
1.127	5.85%	1.1225	5.52%	1.115	6.02%	1.1	6.78%	1.1	7.34%
1.128	5.83%	1.125	5.47%	1.12	5.94%	1.11	6.62%	1.11	7.21%
1.129	5.81%	1.1275	5.43%	1.125	5.86%	1.12	6.47%	1.12	7.08%
1.13	5.80%	1.13	5.40%	1.13	5.80%	1.13	6.35%	1.13	6.97%
1.131	5.79%	1.1325	5.38%	1.135	5.74%	1.14	6.25%	1.14	6.85%
1.132	5.80%	1.135	5.37%	1.14	5.70%	1.15	6.18%	1.15	6.76%
1.133	5.81%	1.1375	5.37%	1.145	5.68%	1.16	6.14%	1.16	6.69%
1.134	5.83%	1.14	5.38%	1.15	5.68%	1.17	6.14%	1.17	6.64%
1.135	5.85%	1.1425	5.39%	1.155	5.69%	1.18	6.18%	1.18	6.61%

- With the current EUR/USD spot = **1.1308**
- And the IRstructure in USD and EUR being:
- The goal is to calibrate the Heston model to this data so that it can be used for the valuation of more complex (exotic) options

Maturity	r_EUR	r_USD
1W	-0.38%	2.42%
1M	-0.37%	2.48%
3M	-0.31%	2.61%
6M	-0.23%	2.68%
12M	-0.11%	2.86%

Source: Investing.com, Author

Heston calibration - Example

- The calibration would proceed as follows:
  1. Compute option prices from the implied volatilities using B-S formula
  2. Set the parameters of the Heston model to some initial values, for example $\kappa = 0.1$, $\theta = 0.1$, $\xi = 0.1$, $\rho = 0.1$ and $V_0 = 0.1\, p.a.$
  3. Use Heston model to compute prices of all options in the term structure
  4. Transform the Heston option prices into B-S implied volatilities
  5. Use appropriate optimization algorithm in order to set the values of the parameters κ, θ, ξ, ρ and V_0 to minimize the sum of squared differences between the market implied volatilities and the implied volatilities from the Heston model based option prices
- The calibration can be done in Matlab, using the functions:
  - blsprice – To compute the Black-Scholes prices
  - optByHestonNI – To calculate Heston option prices
  - blsimpv – To calculate the Black-Scholes implied volatilities
  - fmincon – To perform the optimization

SABR volatility model

- Stochastic alpha, beta, rho (3 parameters) model:
- $dF_t = \sigma_t F_t^{\beta} dW_t$
- $d\sigma_t = \alpha \sigma_t dZ_t$
- $dW_t dZ_t = \rho dt$
- Where F_t is the forward stock price and dW_t and dZ_t are correlated Wiener processes with correlation ρ
- Represents stochastic version of the CEV model with skew given by β and volatility of the volatility by α
- Can be calibrated to fit the volatility smile
- The model has a simple analytical solution that can be expressed in terms of the implied volatility of the Black model (causing it to match the SABR price)

Realized Volatility

- Assume a general SV model $dr = \mu dt + \sigma dz$
- Since $dr^2 = \mu^2 dt^2 + 2\mu\sigma dt dz + \sigma^2 dt$
- We can define integrated variance as

$$IV(t) = \int_{t-1}^{t} \sigma^2(s) ds = \int_{t-1}^{t} dr^2$$

- Empirically approximated by the realized volatility

$$RV(t,\Delta) = \sum_{j=1}^{n} r^2(t-1+j\Delta,\Delta)$$

Realized volatility and jumps

- In presence of jumps the quadratic variance $QV(t)=\int_{t-1}^{t} dr^2$ can be decomposed

$$QV(t)=IV(t)+\sum_{t-1<s\le t,\, dN(s)=1}\kappa^2(s)$$

- Jumps can be filtered out by the realized bi-power variation

$$BV(t,\Delta)=\frac{\pi}{2}\sum_{j=2}^{1/\Delta}|r(t-1+(j-1)\Delta,\Delta)|\times|r(t-1+j\Delta,\Delta)|$$

- And so the jumps can be identified inspecting the difference $RV(t,\Delta)-BV(t,\Delta)$

Z-Estimator of jumps

- The difference $RV(t,\Delta) - BV(t,\Delta)$ is plagued by large estimation noise due to the discreetness of Δ
- This can be quantified by using the integrated quarticity $TQ = \int_{t-1}^{t} \sigma_s^4 ds$, consistently estimated (in the presence of jumps) with the realized tri-power quarticity:

$$\mathrm{TQ}(\mathrm{t},\Delta) = \frac{\pi^{3/2}}{4\Delta}\Gamma\left(\frac{7}{6}\right)^{-3}\sum_{j=3}^{1/\Delta}|\mathrm{r}(\mathrm{t}-1+\mathrm{j}\Delta,\Delta)|^{4/3}|\mathrm{r}(\mathrm{t}-1+(\mathrm{j}-1)\Delta,\Delta)|^{4/3}|\mathrm{r}(\mathrm{t}-1+(\mathrm{j}-2)\Delta,\Delta)|^{4/3},$$

- Statistically significant jumps can then be estimated with the Z-Estimator, which asymptotically follows the standard normal distribution in the days of no jumps:

$$Z(\mathrm{t},\Delta) = \frac{[RV(t,\Delta) - B\mathrm{V}(\mathrm{t},\Delta)]RV(t,\Delta)^{-1}}{\sqrt{[(\pi/2)^2 + \pi - 5]\max\{1, TQ(\mathrm{t},\Delta)B\mathrm{V}(\mathrm{t},\Delta)^{-2}\}\Delta}}$$

$$\mathrm{JV}(\mathrm{t},\Delta) = I\{Z(\mathrm{t},\Delta) > \Phi(\alpha)^{-1}\}[\mathrm{RV}(\mathrm{t},\Delta) - \mathrm{BV}(\mathrm{t},\Delta)]$$

Model Free Volatility

- Expected integrated variance, i.e. $E\left[\int_0^T \sigma_t^2 dt\right]$
- According to Neuberger and Britten-Jones (2000) can be derived from the continuum of option prices $C(T,K)$

$$E_0^F\left[\int_0^T \sigma_t^2 dt\right] = E_0^F\left[\int_0^T \left(\frac{dF_t}{F_t}\right)^2 dt\right] = 2\int_0^\infty \frac{C^F(T,K) - \max(0, F_0 - K)}{K^2} dK$$

- The result holds also in presence of jumps (Jiang and Tian, 2005)

Stochastic-Volatility Jump-Diffusion

- SVJD model class – the most general models
- **Stochastic volatility** – Increases the tails of the return distribution in longer horizons
- **Jumps** – Increase the tails of the return distribution in shorter horizons
- Example – **Log-Variance model with Poisson jumps**
- Log-Price process: $dp(t) = \mu dt + \sigma(t)dz(t) + j(t)dq(t)$
- Log-Variance process: $dh(t) = \kappa[\theta - h(t)]dt + \xi dz_V(t)$
- Where: $h(t) = \ln[\sigma^2(t)]$ $j(t) \sim N(\mu_J, \sigma_J)$ $\Pr[\mathrm{dq(t)} = 1] = \lambda \mathrm{dt}$
- The model can further assume correlation between dz and dz_V, time-variability of λ, or jumps in $h(t)$
- Parameter estimation with MCMC

Model estimation vs. calibration

- There are two ways of how to estimate parameters of stochastic processes used for option pricing:
  1. **Calibration to quoted options** – The parameters of the model are set so that it correctly prices all quotes (typically plain-vanilla) options on the market (i.e. captures the volatility surface). The benefit of the method is that it corresponds to the risk-neutral setting, it is forward looking and can be quick if an analytical formula for the option prices is available. The calibrated model can then be used to price exotic options.
  2. **Estimation on historical data** – The parameters of the model are fitted in order to explain in the best possible way (i.e. maximum likelihood) the historical asset price returns. The main drawback of the method is that the parameters may not correspond to the risk-neutral setting that we use in option pricing. The main benefit is that the model can be assumed to accurately reflect the dynamics of the price process and it can thus be used for computing the expected payoff from the option as well as Value at Risk and Expected shortfall.

MCMC estimation of SVJD models

- Estimation on past historical data is problematic as in addition to the model parameters, we need to estimate the vectors of latent state variables (**V, Q, J**)
- <u>**Markov-Chain Monte-Carlo (MCMC)** method</u>:
- Assume we want to estimate the joint posterior density $p(\Theta|\mathrm{data})$, of all of the model parameters and latent states given by $\Theta = (\theta_1, \dots, \theta_k)$
- MCMC contsructs a Markov Chain, using only the information about the contitional densities $p(\theta_j|\theta_i, i \neq j, \mathrm{data})$, that converges to the target density $\Theta = (\theta_1, \dots, \theta_k)$
- Types of MCMC algorithms: Gibbs Sampler, Metropolis Hastings, Accept-Reject Gibbs, etc.

Gibbs Sampler

- The **Gibbs sampler** proceeds as follows:

1. Assign a vector of initial values to $\Theta^0 = (\theta_1^0, \ldots, \theta_k^0)$ and set $j = 0$
2. Set $j = j + 1$
3. Sample $\theta_1^j \sim p(\theta_1 | \theta_2^{j-1}, \ldots, \theta_k^{j-1}, \text{data})$
4. Sample $\theta_2^j \sim p\left(\theta_2 \middle| \theta_1^j, \theta_3^{j-1}, \ldots, \theta_k^{j-1}, \text{data}\right)$
5. …
6. Sample $\theta_k^j \sim p(\theta_k | \theta_1^j, \theta_2^j, \ldots, \theta_{k-1}^j, \text{data})$ and return to step 1.

- The conditional densities are typically derived from:

$$p\left(\theta_1 \middle| \theta_2^{j-1}, \ldots, \theta_k^{j-1}, \text{data}\right) \propto L\left(\text{data} \middle| \theta_1, \theta_2^{j-1}, \ldots, \theta_k^{j-1}\right) * \text{prior}\left(\theta_1 \middle| \theta_2^{j-1}, \ldots, \theta_k^{j-1}\right)$$

Metropolis-Hastings algorithm

- To utilize the Metropolis-Hastings algorithm, Step 2 in the Gibbs Sampler algorithm has to be replaced by the following two step procedure:

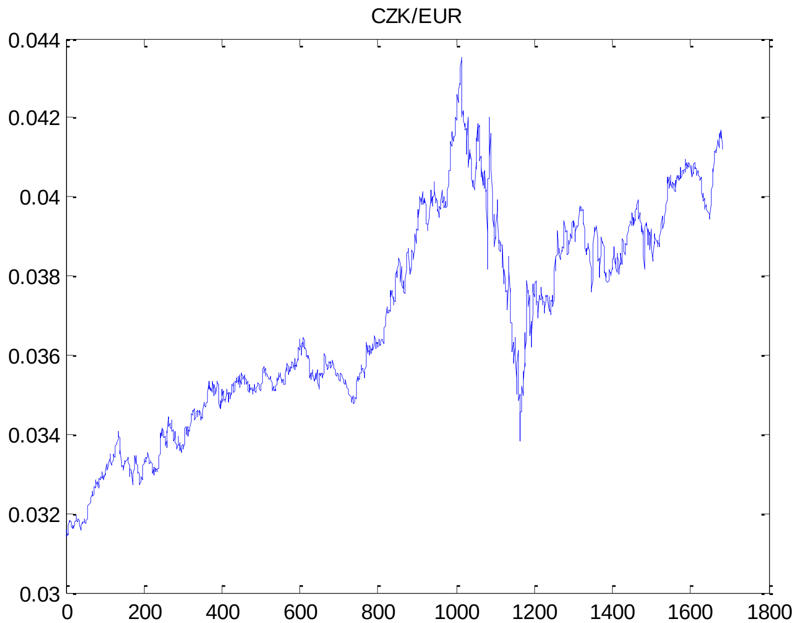
A. Sample θ_1^j from a proposal density $q(\theta_1|\theta_2^{j-1},\dots,\theta_k^{j-1},\text{data})$

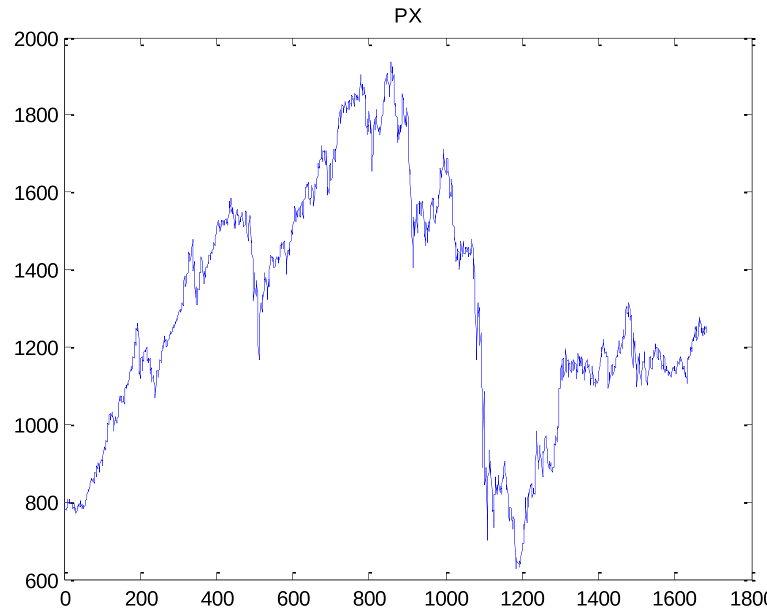
B. Accept θ_1^j with probability $\alpha = \min(R,1)$, with R denoting the so called acceptance ratio:

$$R = \frac{p(\theta_1^j|\theta_2^{j-1},\dots,\theta_k^{j-1},\text{data})q(\theta_1^{j-1}|\theta_1^j,\theta_2^{j-1},\dots,\theta_k^{j-1},\text{data})}{p(\theta_1^{j-1}|\theta_2^{j-1},\dots,\theta_k^{j-1},\text{data})q(\theta_1^j|\theta_1^{j-1},\theta_2^{j-1},\dots,\theta_k^{j-1},\text{data})}$$

Analysis of Jumps and Stochastic Volatility for EUR/CZK a PX

- Data 2/9/2004 – 11/2/2011





Source: https://pep.vse.cz/pdfs/pep/2013/02/07.pdf

Estimation of jump-diffusion model parameters

- Discrete model

$$r_i = \mu + \sigma \check{n}_i + Z_i J_i$$

$$\check{n}_i \sim N(0,1),\, Z_i \sim N(\mu_J, \sigma_J),\, J_i \sim \mathrm{Bern}(\lambda), \mathrm{iid}$$

- MCMC estimated (simulated) variables and parameters: $\mu, \sigma, \lambda, \mu_J, \sigma_J, \mathbf{Z}, \mathbf{J}$
- In the case of a jump-diffusion model state variables (jump times and sizes) do not have to be necessarily estimated, but there is a model consistent identification of jumps as a side product

Empirical Results

CZK/EUR (daily returns)

μ	σ	λ	μ_J	σ_J
2.3842e-004 (9.0573e-005)	0.0030 (1.0443e-004)	0.1849 (0.0260)	-8.5108e-005 (5.9265e-004)	0.0093 (6.6180e-004)

Source: Author

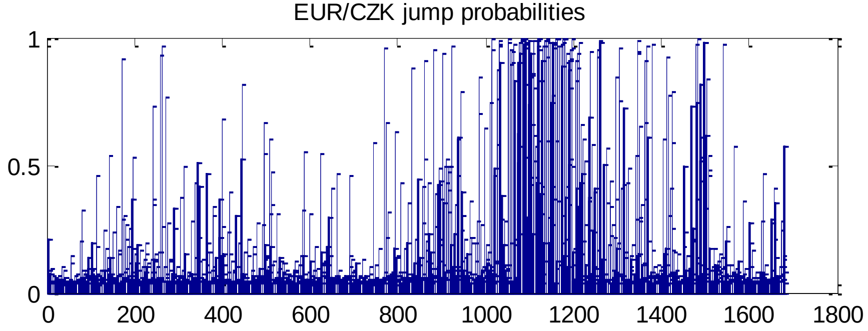
PX (daily returns)

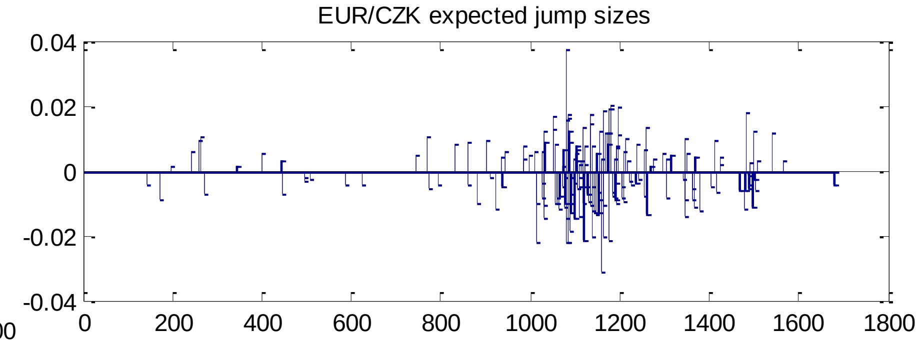
μ	σ	λ	μ_J	σ_J
0.0010 (2.8714e-004)	0.0101 (3.6742e-004)	0.1530 (0.0227)	-0.0041 (0.0025)	0.0355 (0.0026)

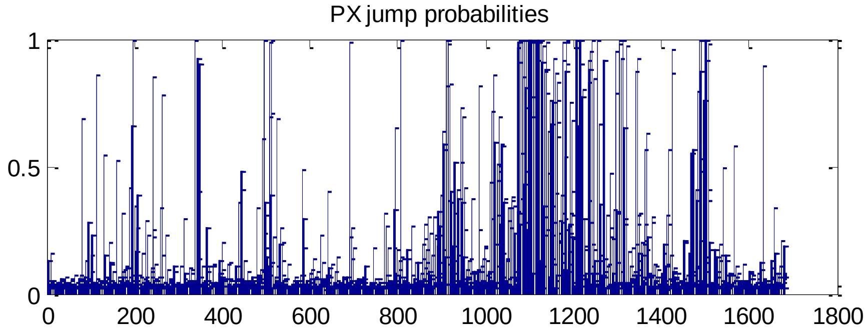
Source: Author

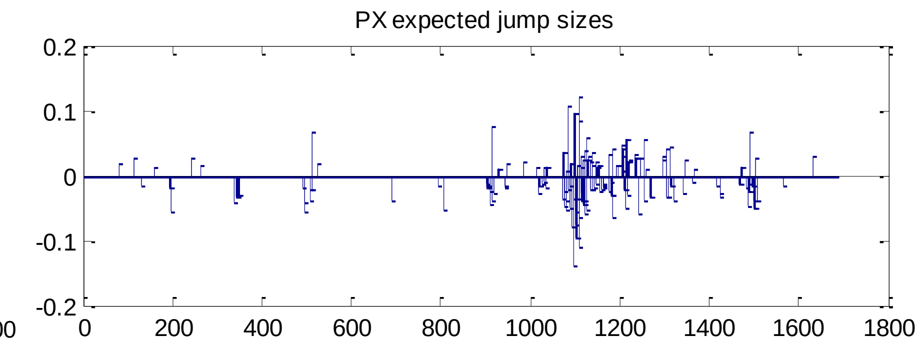
Jump Times and Sizes

- Average jump size is shown only for times, where P[J=1]>0,5
- It is obvious that there is a jumps clustering







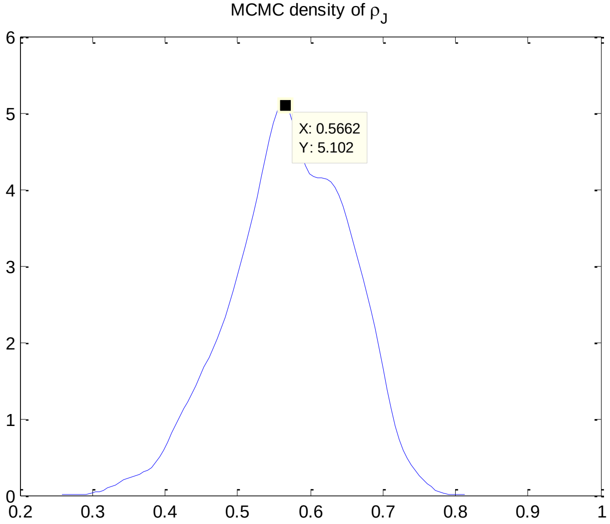


Source: https://pep.vse.cz/pdfs/pep/2013/02/07.pdf

Extension of the model with correlations

- MCMC process can be implemented simultaneously for both time series estimation correlations between ε, Z a J
- Diffusion and jump size correlations are negligible but there is a significant correlation between jump occurrence

ρ_D	ρ_Z	ρ_J
-0.0444 (0.0345)	0.0052 (0.0936)	0.5549 (0.1106)



Source: Author

Jump-diffusion model with stochastic volatility

- It seems that volatility clustering corresponds to crisis periods with large volatility, and so the model should be extended with stochastic volatility

$$r_i = \mu + \sqrt{V_i}\varepsilon_i + Z_i J_i$$

$$\log V_i = \alpha + \beta \log V_{i-1} + \gamma \varepsilon_i^V$$

$$\varepsilon_i, \varepsilon_i^V \sim N(0,1),\ Z_i \sim N(\mu_J, \sigma_J),\ J_i \sim \text{Bern}(\lambda), \text{iid}$$

- MCMC process must be extended with a variance vector $\mathbf{V}$ whose estimation is non-trivial (Metropolis)

Test of the Model

Returns (T=2000) generated with the parameters:

μ	λ	μ_J	σ_J	α	β	γ
0.01	0.03	0.03	0.11	-0.14	0.98	0.15

Source: Author

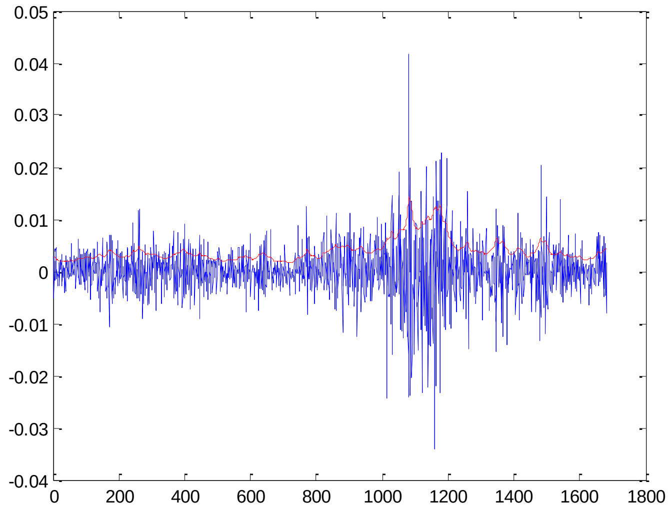
MCMC estimations

μ	λ	μ_J	σ_J	α	β	γ
0.0091 (5.6548e-004)	0.0391 (0.0108)	0.0370 (0.0181)	0.1092 (0.0126)	-0.1821 (0.0884)	0.9731 (0.0131)	0.1346 (0.0367)

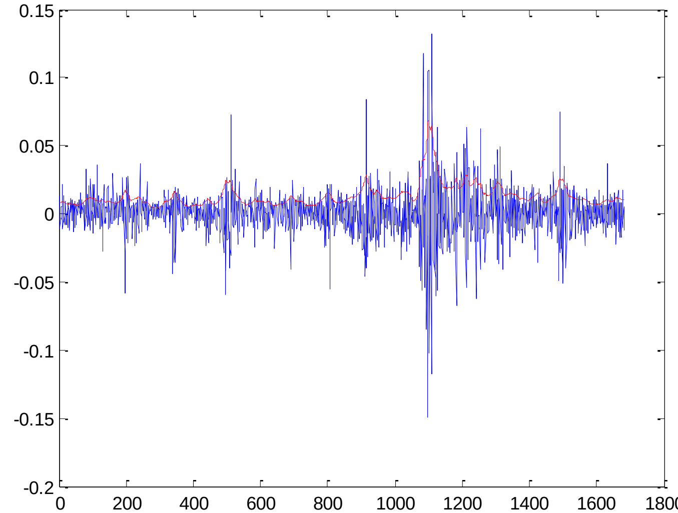
Source: Author

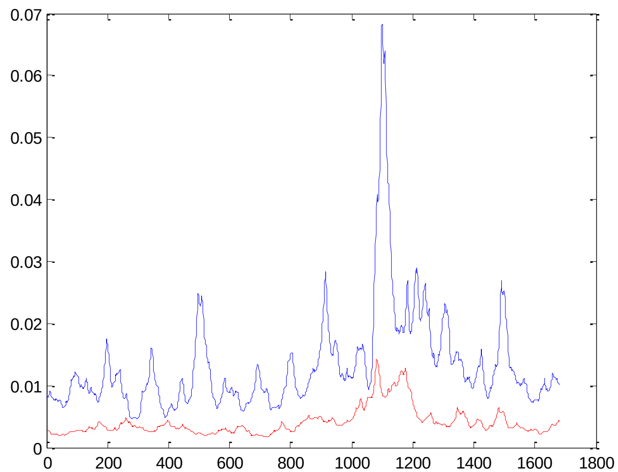
Univariate stochastic volatility models estimates

CZK/EUR



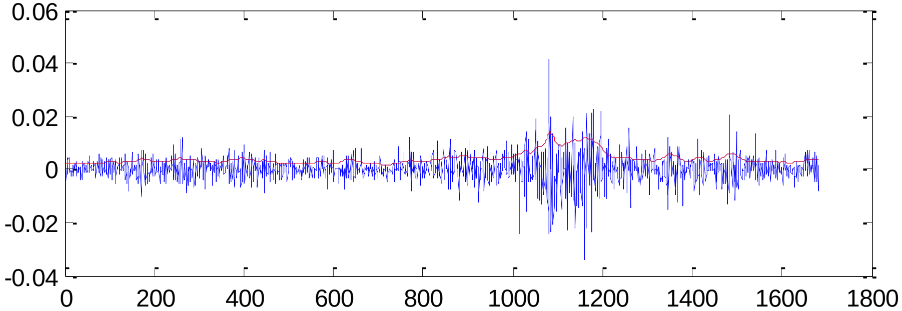
PX



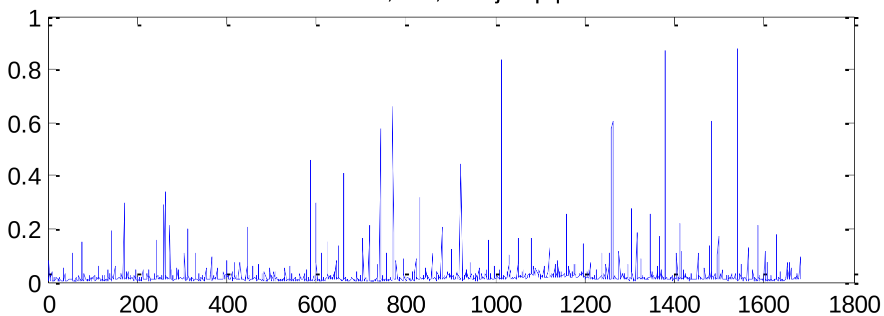


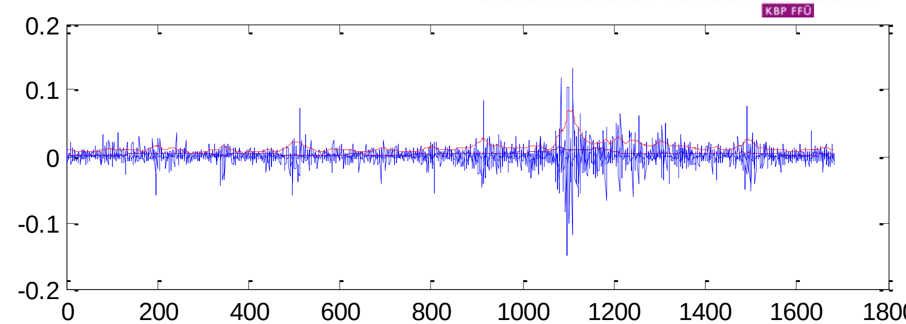
There is a significant reduction of jumps (to 2-3%) and their correlation. On the other there is a high correlation between the stochastic Volatilities levels (over 50%)

Source: https://pep.vse.cz/pdfs/pep/2013/02/07.pdf, Author

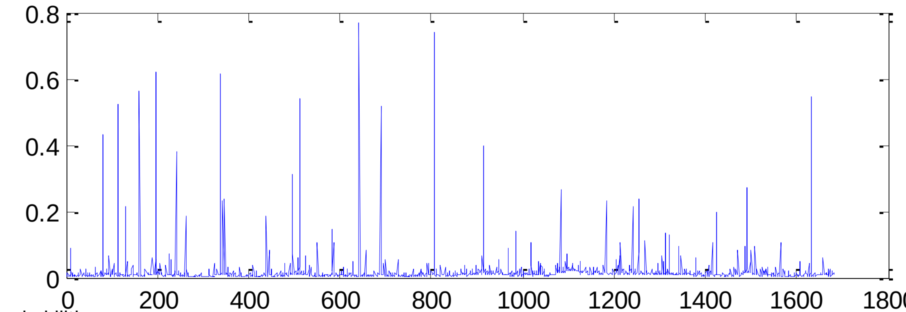


CZK/EUR return, SV, and jump probabilities

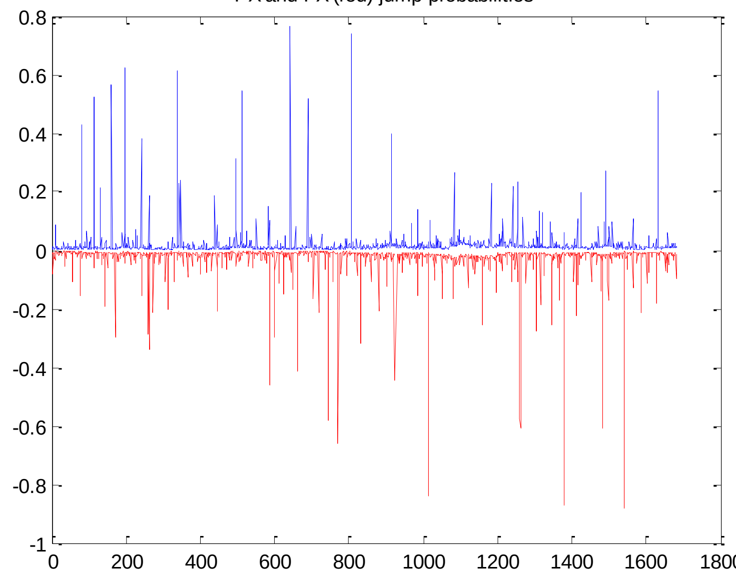




PX returns, SV, and jump probabilities



PX and FX (red) jump probabilities



Source: https://pep.vse.cz/pdfs/pep/2013/02/07.pdf, Author

Empirical Results

CZK/EUR (daily returns)

μ	λ	μ_J	σ_J	α	β	γ
1.8506e-004(6.3746e-005)	0.0284(0.0083)	-2.2616e-004(0.0024)	0.0117(0.0018)	-0.1205 (0.0545)	0.9893 (0.0048)	0.1313 (0.0193)

Source: Author

PX (daily returns)

μ	λ	μ_J	σ_J	α	β	γ
0.0012 (1.9213e-004)	0.0237 (0.0068)	0.0011(0.0079)	0.0427 (0.0066)	-0.1957 (0.0613)	0.9781 (0.0069)	0.2119 (0.0247)

Source: Author

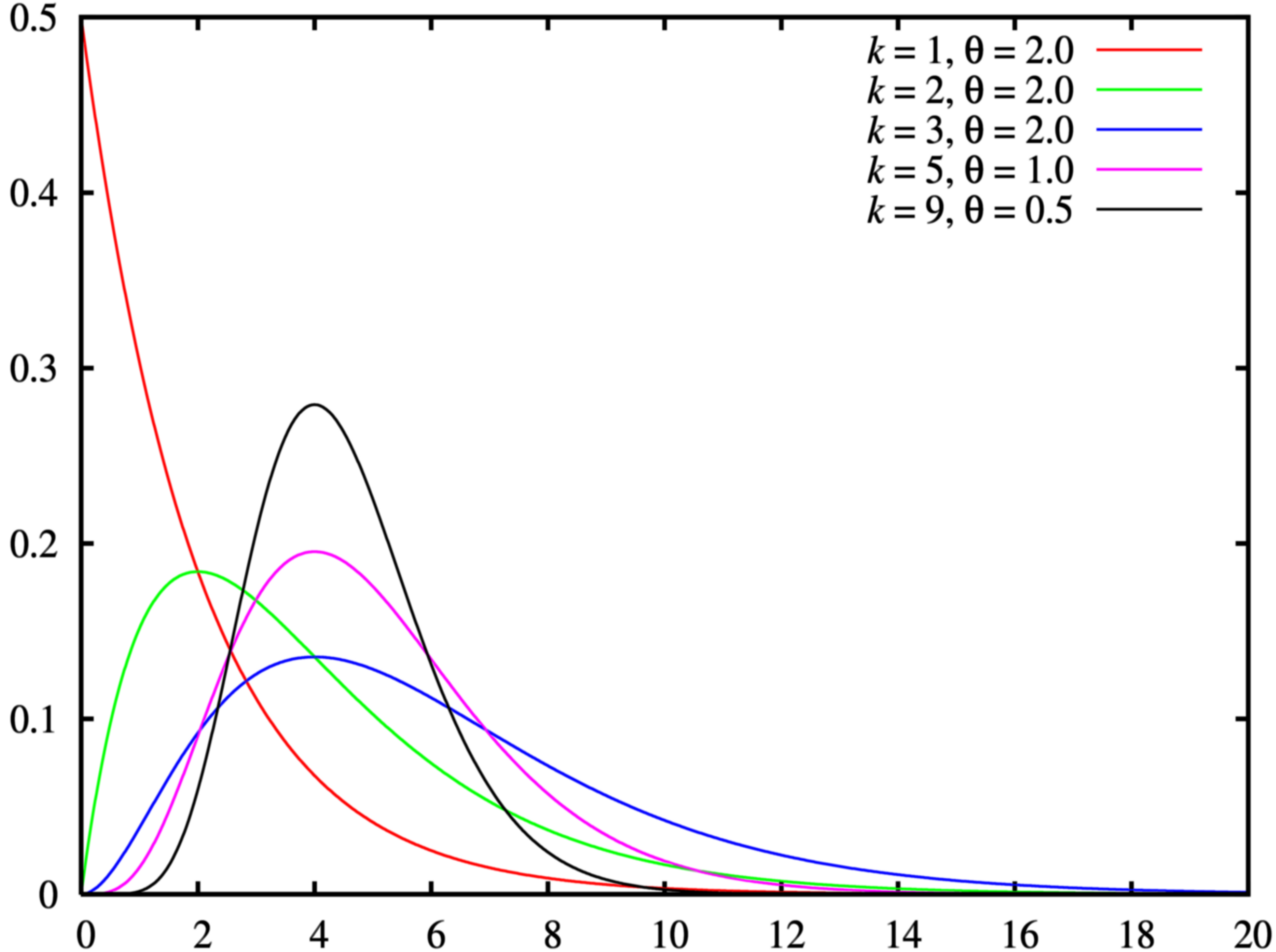
Particle filters

- Assume we have an observed time series y_t and observable series x_t, where: $$y_t \sim p(y_t|x_t,\theta)$$ $$x_t \sim p(x_t|x_{t-1},\theta)$$
- The goal of the filtering problen is to estimate $p(x_t|y_{1:t},\theta)$
- The SIR Particle Filter (Gordon, 1993) proceeds as follows:
  1. We represent the density $p(x_{t-1}|y_{1:t-1},\theta)$ with a weighted set of $\mathrm{i} = 1,\dots,M$ particles $x_{t-1}^{(i)}$ with weights $\tilde{w}_{t-1}^{(i)}$
  2. We simulate new particles for time t from a proposal density $g(x_t|x_{t-1},y_t)$
  3. We compute the weights for time t with: $w_t^{(i)} = \frac{p(y_t|x_t^i)p(x_t^i|x_{t-1}^i)}{g(x_t^i|x_{t-1}^i,y_t)}\tilde{w}_{t-1}^{(i)}$
  4. We normalize the weights: $\tilde{w}_t^{(i)} = w_t^{(i)}/\sum_{j=1}^{M} w_t^{(i)}$
  5. If $ESS = 1/\sum_{i=1}^{M}\left(\tilde{w}_t^{(i)}\right)^2 < ESS_{Thr}$ we re-sample the particles with probability of being sampled equal to $\tilde{w}_t^{(i)}$, and we set all of the weights to $\tilde{w}_t^{(i)} = 1/M$

Variance-Gamma Model

- The idea is that the future price development depends on the information flow rather than on time itself
- To model S_T we firstly generate “the information time” g using the Gamma distribution and then S_T as a lognormal variable with variance $\sigma^2 g$ and with an appropriate mean
- Additional parameters: v… the variance rate of the gamma process, θ…skewness
- Tends to produce U-shaped volatility smiles

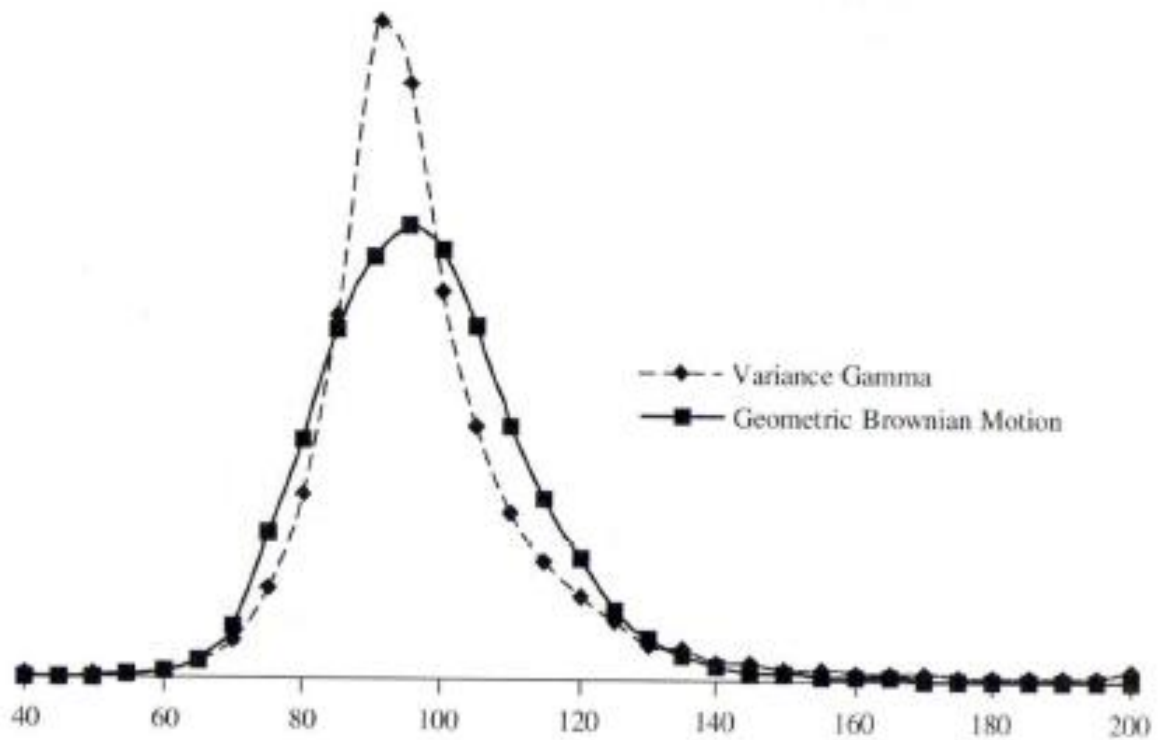
Gamma Distribution



Source: http://smartdrill.com/images/Gamma%20distributions.jpg

Variance Gamma Process

Distributions obtained with variance-gamma proces and geometrie Brownian motion.



Source: Author

Option Valuation with Fast Fourier Transform

- Carr and Madan (1999) proved that for exponential Lévy processes it is possible to express the value of a plain-vanilla call option by using the Fast Fourier Transform and the characteristic function of the process as follows:

$$C_0(T,K) = \frac{e^{-\varrho \ln K}}{\pi} \int_{-\infty}^{+\infty} e^{-i\theta \ln K} \frac{e^{-rT} \varphi^{Q}_{\ln S_T}\left(\theta - i(1+\varrho)\right)}{\varrho^2 + \varrho - \theta^2 + i\theta(2\varrho + 1)} d\theta,$$

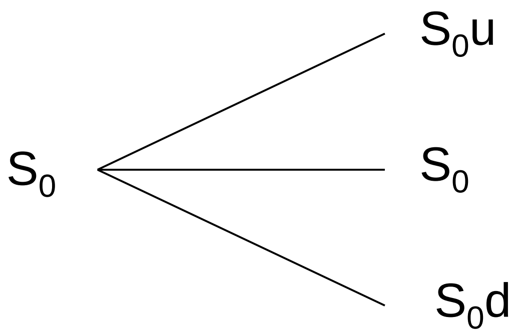
- Where $\varphi^{Q}_{\ln S_T}$ denotes the characteristic function of the logarithm of he risk-neutral process, describing the dynamics of the underlying asset at time T
- Use of the formula greatly simplifies the calibration of many of the models mentioned in previous sections

Content

- ✓ Convexity, time, and quanto adjustments
- ✓ Short-rate and advanced interest rate models
- ✓ Volatility smiles
- ✓ Exotic options
- ✓ Alternative stochastic models
- ➢ Numerical methods for option pricing
- Credit derivatives

Numerical Methods

- **Binomial trees** – useful in particular for valuation of American options working backward through the tree
- Control variate technique estimates an American option as $f_A+(f_{BS}-f_E)$
- **Trinomial trees** can be used as an alternative, in particular for barrier options and interest rate derivatives



Source: Author

Monte Carlo Simulations

- Appropriate for path-dependent options (e.g. Asian)
- Generally time consuming, if ω is the standard deviation of the variable being estimated and M the number of steps then the standard error is $\frac{\omega}{\sqrt{M}}$
- There are, however, various variance reduction techniques (antithetic variable, control variate, importance sampling, stratified sampling, moment matching, quasi-random sequences, ..) leading to an improvement up to the order of $\frac{\omega}{M}$

Other Numerical Techniques

- Finite difference methods to solve partial differential equations
- Binomial Trees for path dependent derivatives
- Binomial Trees in two or more dimension with a correlation
- Monte Carlo simulations for American options
- See Chapters 20 and 26 in Hull, 8th Edition

Content

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- ➢ Credit derivatives

Literature

FINANCIAL ENGINEERING KBP FFÚ

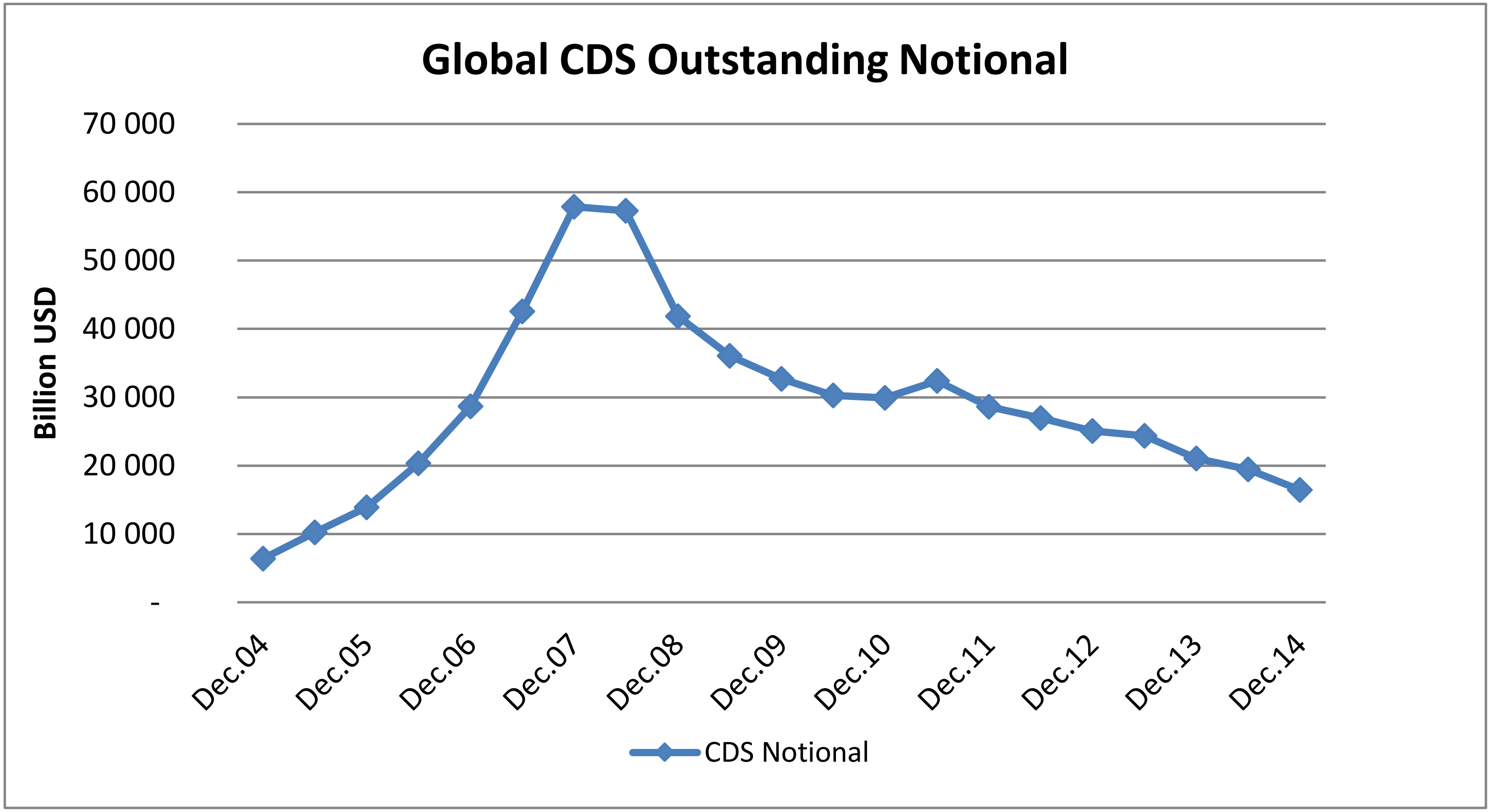
Requirement	Title	Author	Year of Publication
Required	Credit Risk Management and Modeling	Witzany, J.	2010, Oeconomica, pp. 215
Optional	Managing Credit Risk – The Great Challenge for Global Financial Markets	Caouette J.B., Altman E.I., Narayan O., Nimmo R.	2008, 2nd Edition, Wiley Finance, pp. 627
Optional	Credit Risk – Pricing, Measurement, and Management	Duffie D., Singleton K.J.	2003, Princeton University Press, pp.396
Optional	Consumer Credit Models: Pricing, Profit, and Portfolios	Thomas L. C.	2009, Oxford University Press, pp. 400
Optional	Credit Derivatives Pricing Models	Schönbucher P.J.	2003, Wiley Finance Series, pp. 375

Source: Author

Credit Derivatives

- Payoff depends on creditworthiness of one or more subjects
- Single name or multi-name
- Credit Default Swaps, Total Return Swaps, Asset Backed Securities, Collateralized Debt Obligations
- Banks – typical buyers of credit protection, insurance companies – sellers

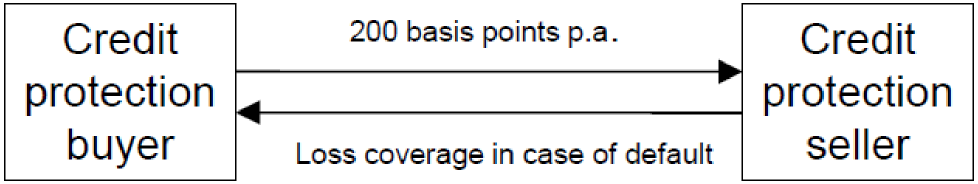
Credit Default Swaps



Source: Author

Credit Default Swaps

- CDS spread usually paid in arrears quarterly until default
- Notional, maturity, definition of default
- Reference entity (single name)
- Physical settlement – protection buyer has the right to sell bonds (CTD)
- Cash settlement – calculation agent, or binary
- Can be used to hedge corresponding bonds



Source: Author

Valuation of CDS

- Requires risk neutral probabilities of default for all relevant maturities
- Market value of a CDS position is then based on the general formula

$$MV = E_Q[\text{discounted cash flow}] = \sum_{i=1}^{n} e^{-r_i T_i} E_Q[\text{cash flow}_i]$$

- Market equilibrium CDS spread is the spread that makes *MV*=0

Risk neutral default probabilities

- Can be calculated from bond prices using the general formula and bootstrapping
- Example: Given bond prices for maturities 1,2,3 calculate the probabilities (LGD=0.4)

Bond Value	Coupon	Maturity	R	Q
101,00	3,50	1	2,00%	1,11%
102,50	5,00	2	3,00%	3,20%
102,00	5,00	3	3,50%	5,45%

Source: Author

$$101 = e^{-0.02} \cdot 103.5 \cdot 0.6 + e^{-0.02} \cdot (1 - Q_1) \cdot 103.5 \cdot 0.4$$

$$102.5 = e^{-0.02} \cdot 5 \cdot 0.6 + e^{-0.03} \cdot 105 \cdot 0.6 + e^{-0.02} \cdot (1 - 0.0111) \cdot 5 \cdot 0.4 + e^{-0.03} \cdot (1 - Q_2) \cdot 105 \cdot 0.4$$

Default intensities

- In order to interpolate/extrapolate the cumulative PDs it is useful to work with default intensities, i.e. hazard rates

$$\lambda(t) = \frac{dQ(t)}{1-Q(t)}\frac{1}{dt}, \text{ i.e. } \frac{dS(t)}{dt} = -\lambda(t)S(t)$$

$$Q(t) = 1 - e^{-\int_0^t \lambda(s)ds} = 1 - e^{-\bar{\lambda}(t)t} \qquad \bar{\lambda}(t) = \frac{1}{t}\int_0^t \lambda(s)ds$$

- Example:

Maturity	Q	Aver λ	Annual λ
1	1,11%	1,12%	1,12%
2	3,20%	1,62%	2,13%
3	5,45%	1,87%	2,36%

Source: Author

$$\bar{\lambda}(2.5) = (\bar{\lambda}(2) + \bar{\lambda}(3))/2 = 1.75\% \qquad Q(2.5) = 4.27\%$$

or $$Q(2.5) = 1 - e^{-\lambda_1 - \lambda_2 - 0.5\lambda_3} = 4.33\%$$

Historical versus risk-neutral probabilities

Rating	Historical default intensity	Risk Neutral Default intensity	Ratio	Difference
Aaa	0.04	0.60	16.7	0.56
Aa	0.05	0.74	14.6	0.68
A	0.11	1.16	10.5	1.04
Baa	0.43	2.13	5.0	1.71
Ba	2.16	4.67	2.2	2.54
Caa and lower	13.07	18.16	1.4	5.5

Source: Author

Valuation of single name CDS

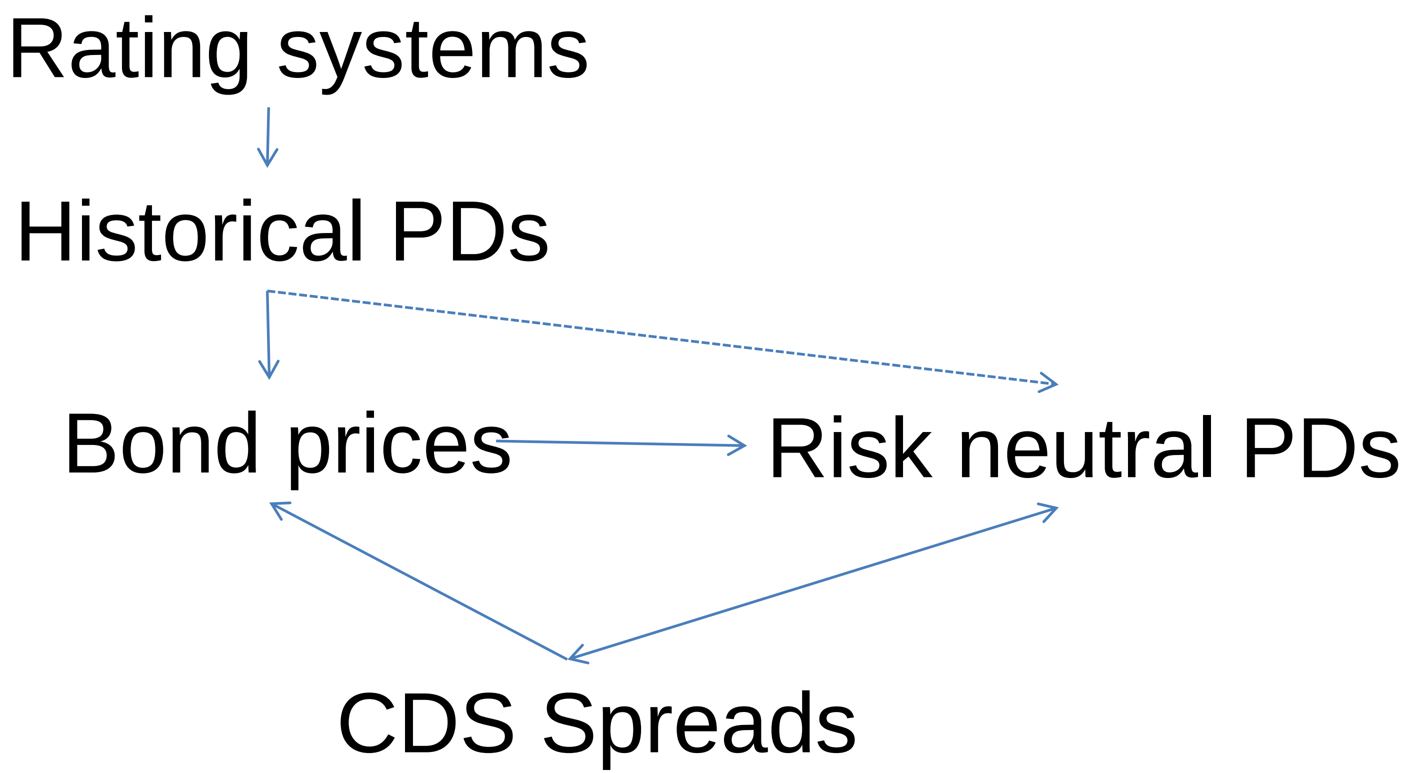
- 3Y CDS, we pay 120 bps on $100, calculate the market value given the probabilities obtained above, LGD=0.4, and assuming that defaults can happen only halfway through a year

Time	Q	R	Cash Flow	Probability	Expected PV
0,5		2,00%	39,40	1,11%	0,43
1	1,11%	2,00%	- 1,20	98,89%	- 1,16
1,5		2,50%	39,40	2,09%	0,79
2	3,20%	3,00%	- 1,20	96,80%	- 1,09
2,5		2,17%	39,40	2,26%	0,84
3	5,45%	3,50%	- 1,20	94,55%	- 1,02
Total					**- 1,21**

Source: Author

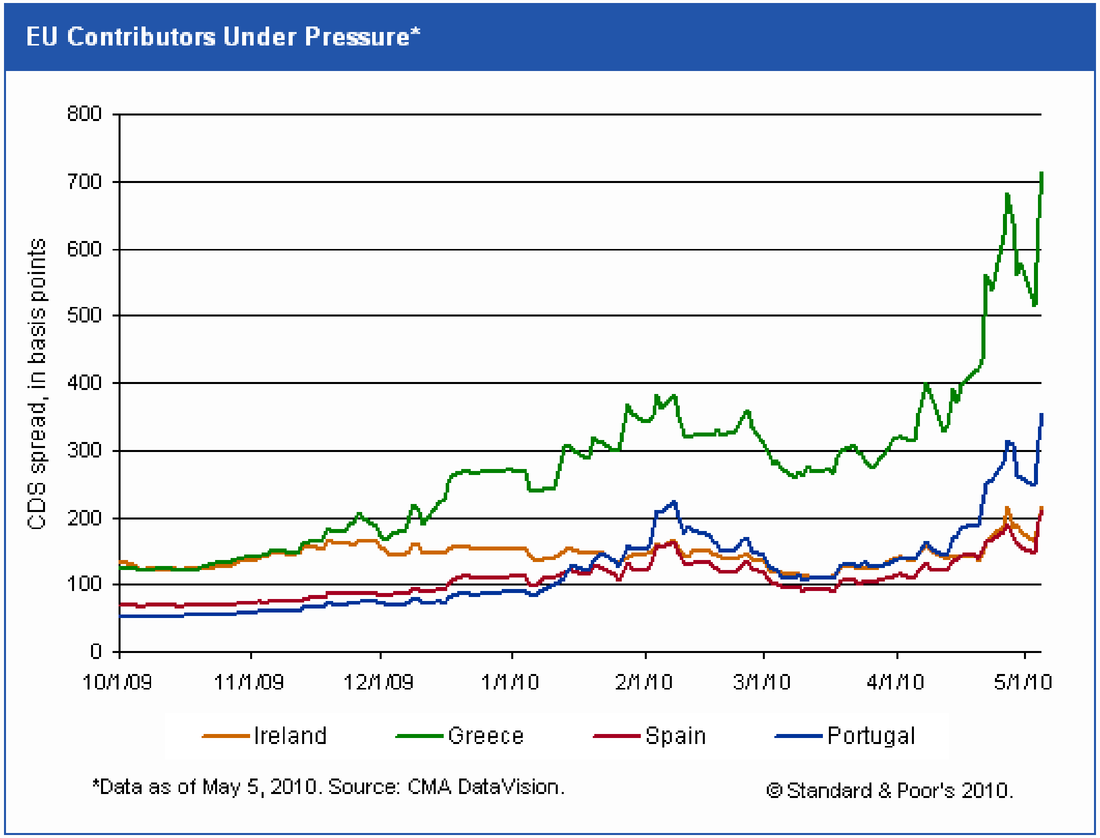
- Market spread of appr. 76 bps makes MV=0

Estimating Default Probabilities



Source: Author

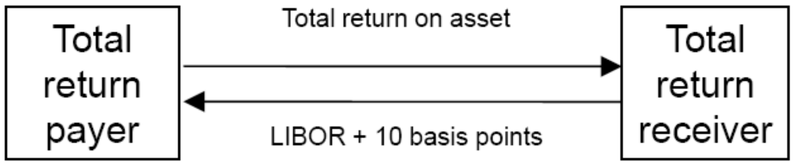
Example of CDS quotes



Source: Standard & Poor`s, 2010

Total Return Swap

- Total return payer gives up the return including the credit risk spread and receives essentially the risk free return (plus the counterparty credit risk margin)



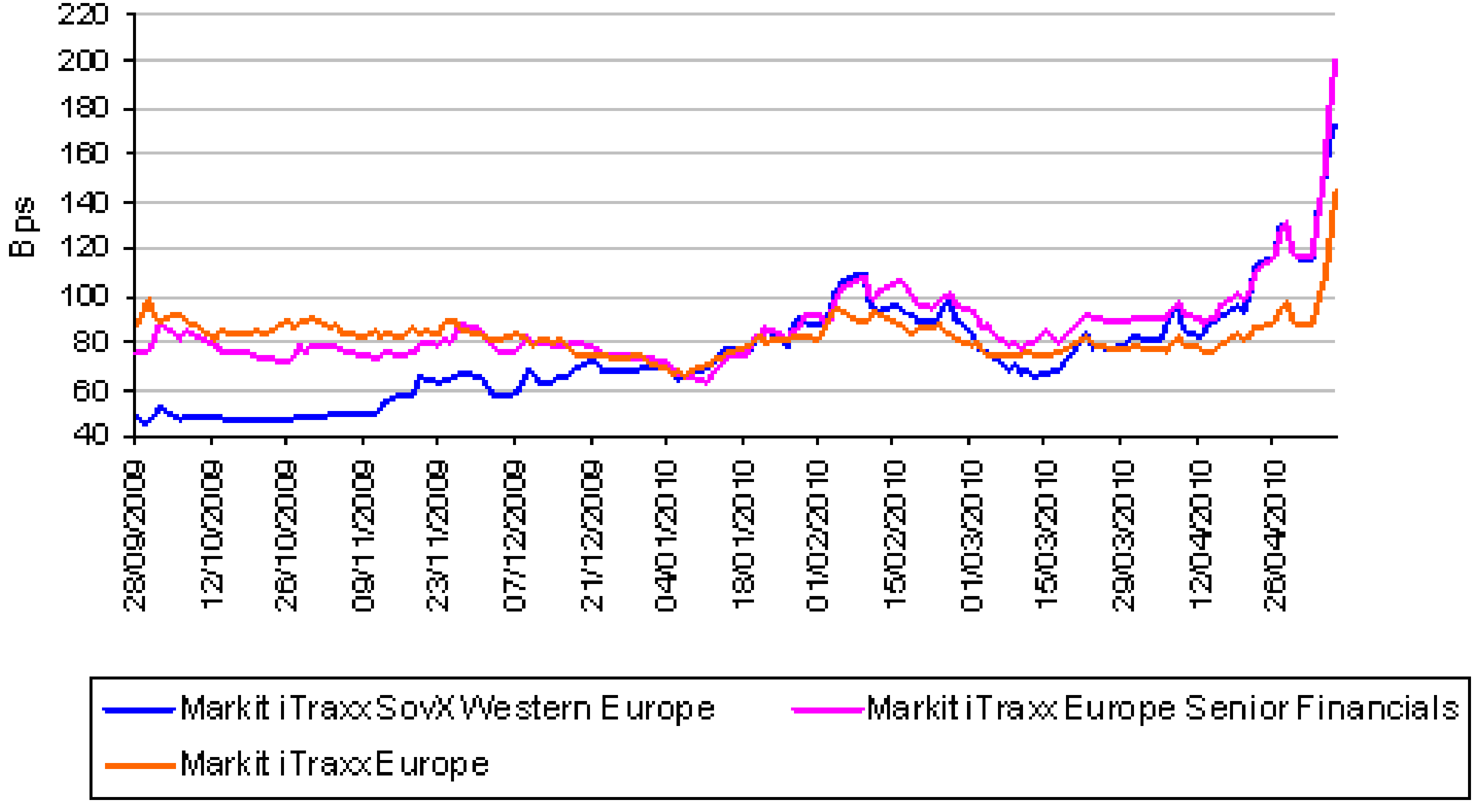
Source: Author

- Total return swap should not be mistaken with the Asset Swap where a risky fix coupon bond investment cash flow is transformed to risky floating coupon par investment cash flow

Credit Indices

- In principle averages of single name CDS spreads for a list of companies
- In practice traded multi-name CDS
- CDX NA IG – 125 investment grade companies in N.America
- iTraxx Europe - 125 investment grade European companies
- Standardized payment dates, maturities (3,5,7,10), and even coupons – market value initial settlement

Credit Indices - Evolution



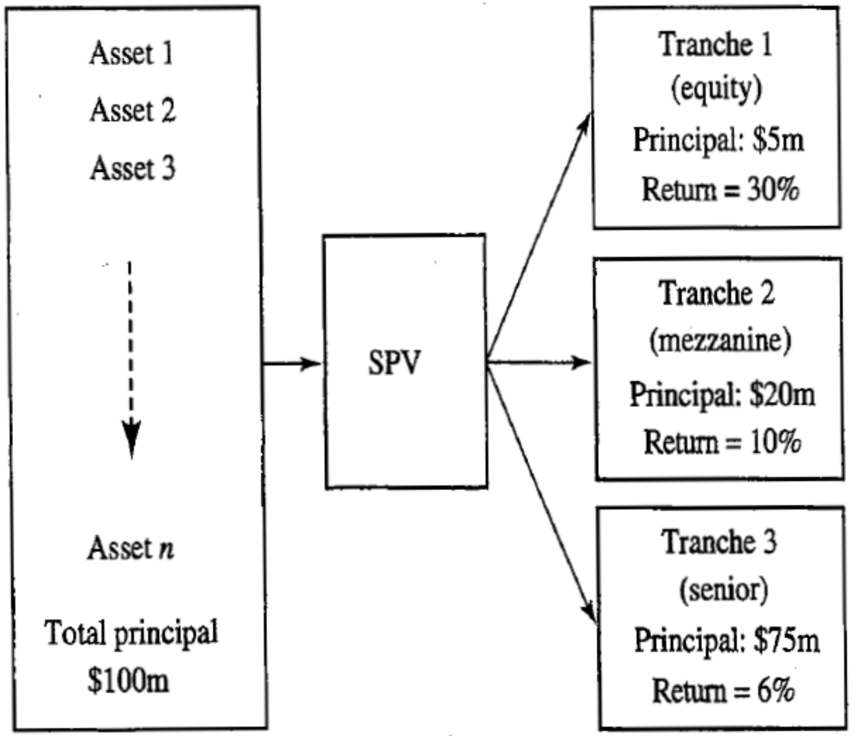
Source: http://1.bp.blogspot.com/_9cc9B-U-py0/S-UyECW5JKI/AAAAAAAABL4/f_yFN9rkE18/s1600/Markit.gif

CDS Forwards and Options Basket CDS

- Defined similarly to forwards and options on other assets or contracts, e.g. IRS
- Valuation of forwards can be done just with the term structure of risk neutral probabilities
- But valuation of options requires a stochastic modeling of probabilities of default (or intensities – hazard rates)
- Many different types of basket CDS: add-up, first-to-default, k-th to default … requires credit correlation modeling

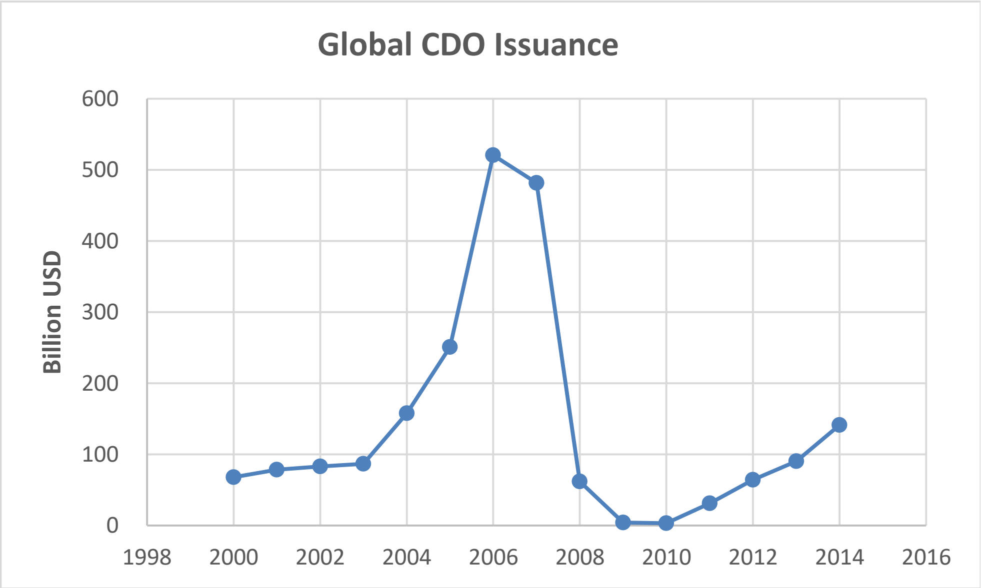
Asset Backed Securities (CDO,..)

- Allow to create AAA bonds from a portfolio of poor assets



Source: Author

CDO market



Year	High Yield Bonds	High Yield Loans	Investment Grade Bonds	Mixed Collateral	Other	Other Swaps	Structured Finance	Total
2000	11 321	22 715	29 892	2 090	932		1 038	**67 988**
2001	13 434	27 368	31 959	2 194	2 705		794	**78 454**
2002	2 401	30 388	21 453	1 915	9 418		17 499	**83 074**
2003	10 091	22 584	11 770	22	6 947	110	35 106	**86 630**
2004	8 019	32 192	11 606	1 095	14 873	6 775	83 262	**157 821**
2005	1 413	69 441	3 878	893	15 811	2 257	157 572	**251 265**
2006	941	171 906	24 865	20	14 447	762	307 705	**520 645**
2007	2 151	138 827	78 571		1 722	1 147	259 184	**481 601**
2008		27 489	15 955				18 442	**61 887**
2009		2 033	1 972				331	**4 336**
2010		1 807	4 806		321		1 731	**8 666**
2011		20 002	1 028		8 126		1 975	**31 131**
2012		44 062	62				20 246	**64 371**
2013	-	26 362	-	-	-	-	63 911	**90 273**
2014	-	70 018	430	-	-	-	70 846	**141 294**

Source: Author

Cash versus Synthetic CDOs

- Synthetic are created using CDS

BY ISSUANCE TYPE ($MM)

	TOTAL ISSUANCE	Cash Flow and Hybrid[2]	Synthetic Funded[3]	Market Value[4]	Arbitrage[5]	Balance Sheet[6]	Long Term[7]	Short Term[8]
2004-Q1	24,982.5	18,807.8	6,174.7	0.0	23,157.5	1,825.0	20,495.1	4,487.4
2004-Q2	42,861.6	25,786.7	17,074.9	0.0	39,715.5	3,146.1	29,611.4	13,250.2
2004-Q3	42,086.6	36,106.9	5,329.7	650.0	38,207.7	3,878.8	34,023.9	8,062.7
2004-Q4	47,487.8	38,829.9	8,657.9	0.0	45,917.8	1,569.9	38,771.4	8,716.4
2004 TOTAL	**157,418.5**	**119,531.3**	**37,237.2**	**650.0**	**146,998.5**	**10,419.8**	**122,901.8**	**34,516.7**
2005-Q1	49,610.2	40,843.9	8,766.3	0.0	43,758.8	5,851.4	45,175.2	4,435.0
2005-Q2	71,450.5	49,524.6	21,695.9	230.0	62,050.5	9,400.0	65,043.6	6,406.9
2005-Q3	52,007.2	44,253.1	7,754.1	0.0	49,636.7	2,370.5	48,656.3	3,350.9
2005-Q4	98,735.4	71,604.3	26,741.1	390.0	71,957.6	26,777.8	88,763.5	9,971.9
2005 TOTAL	**271,803.3**	**206,225.9**	**64,957.4**	**620.0**	**227,403.6**	**44,399.7**	**247,638.6**	**24,164.7**
2006-Q1	108,012.7	83,790.1	24.222.6	0.0	101,153.6	6,859.1	104,084.0	3,928.7
2006-Q2	124,977.9	97,260.3	24,808.4	2,909.2	102,564.6	22,413.3	119,986.1	4,991.8
2006-Q3	138,628.7	102,167.4	14,703.8	21,757.5	125,945.2	12,683.5	135.928.5	2,700.2
2006-Q4	180,090.3	131,525.1	25,307.9	23,257.3	142,534.3	37,556.0	180,090.3	0.0
2006 TOTAL	**551,709.6**	**414,742.9**	**89,042.7**	**47,924.0**	**472,197.7**	**79,511.9**	**540,088.9**	**11,620.7**
2007-Q1	186,467.6	140,319.1	27,426.2	18,722.3	156,792.0	29,675.6	181,341.2	5,126.4
2007-Q2	175,939.4	135,021.4	8,403.0	32,515.0	153.385.4	22,554.0	167,459.2	8,480.2
2007-Q3	93,063.6	56,053.3	5,198.9	31,811.4	86,331.4	6,732.2	90,710.0	2,353.6
2007-Q4	47,508.2	31,257.9	5,202.3	11,048.0	39,593.7	7,914.5	47,508.2	0.0
2007 TOTAL	**502,978.8**	**362,651.7**	**46,230.4**	**94,096.7**	**436,102.5**	**66,876.3**	**487,018.6**	**15,960.2**
2008-Q1**	19,470.7	11,930.1	513.7	7,026.9	18,111.8	1,358.9	19,470.7	0.0
2008-Q2	17,336.7	14,260.4	698.5	2,377.8	10,743.7	6,593.0	17,336.7	0.0
2008 YTD TOTAL	**36,807.4**	**26,190.5**	**1,212.2**	**9,404.7**	**28,855.5**	**7,951.9**	**36,807.4**	**0.0**

Source: Author

Single Tranche Trading

- Synthetic CDO tranches based on CDX or iTraxx

Table 23.6 Five-year CDX NA IG and iTraxx Europe tranches on March 28, 2007. Quotes are 30/360 in basis points except for 0%–3% tranche, where the quote indicates the percent of the tranche principal that must be paid up front in addition to 500 basis points per year.

CDX NA IG						
Tranche	0–3%	3–7%	7–10%	10–15%	15–30%	30–100%
Quote	26.85%	103.8	20.3	10.3	4.3	2.0
iTraxx Europe						
Tranche	0–3%	3–6%	6–9%	9–12%	12–22%	22–100%
Quote	11.25%	57.7	14.4	6.4	2.6	1.2

Source: John Hull, Option, Futures, and Other Derivatives, 7th Edition

Valuation of CDOs

- Sources of uncertainty: times of default of individual obligor and the recovery rates (assumed deterministic in a simplified approach)
- Everything else depends on the Waterfall rules (but in practice often very complex to implement precisely)

Valuation of CDOs

- Monte Carlo simulation approach:
- In one run simulate the times to default of individual obligors in the portfolio using risk neutral probabilities and appropriate correlation structure
- Generate the overall cash flow (interest and principal payments) and the cash flows to individual tranches
- Calculate for each tranche the mean (expected value) of the discounted cash flows

Valuation of CDO and distribution of losses

- Thresholds $0 = x_0 < x_1 < \cdots < x_n = 1$
- Loss $X \in [0,1]$ and

$$\text{Payoff}_i = \begin{cases} 0 \text{ if } X \le x_{i-1} \\ X - x_{i-1} \text{ if } x_{i-1} < X \le x_i \\ x_i - x_{i-1} \text{ if } x_i < X \end{cases}$$

- Tranche *i* value

$$f_i = e^{-rT} E[\text{Payoff}_i] =$$

$$= e^{-rT}\left(\left(F(x_i) - F(x_{i-1})\right) E[X - x_{i-1} \mid x_{i-1} < X \le x_i] + \left(1 - F(x_i)\right)\left(x_i - x_{i-1}\right)\right)$$

- Assumming partial linearity of the loss cdf *F* the up-front spread

$$s_i = \frac{f_i}{x_i - x_{i-1}} \cong e^{-rT}\left(1 - F\left(\frac{x_{i-1} + x_i}{2}\right)\right)$$

Gaussian Copula of Time to Default

- In order to model portfolio loss distribution we need a correlation model
- Guassian Copula is the approach when correlation is modeled on the standard normal transformation of the time to default

$$X_j = N^{-1}(Q_j(T_j))$$

$$X_j = \sqrt{\rho} \cdot M + \sqrt{1-\rho} \cdot Z_j$$

- The single factor approach can be used to obtain an analytical valuation
- Generally used also in simulations

Implied Correlation

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- Correlations implied by market quotes based on the standard one factor model (similarly to implied volatility)

Table 23.8 Implied correlations for 5-year iTraxx Europe tranches on March 28, 2007.

Compound correlations					
Tranche	0–3%	3–6%	6–9%	9–12%	12–22%
Quote	18.3%	9.3%	14.3%	18.2%	24.1%
Base correlations					
Tranche	0–3%	0–6%	0–9%	0–12%	0–22%
Quote	18.3%	27.3%	34.9%	41.4%	58.1%

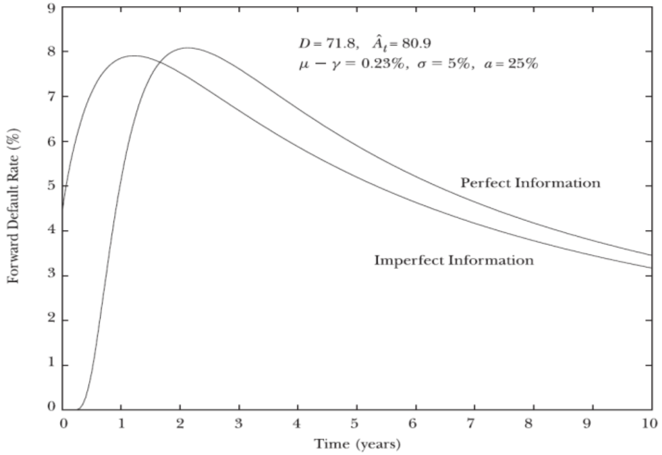
John Hull, Option, Futures, and Other Derivatives, 7th Edition

Alternative Models

- The correlations are uncertain!
- The Gaussian correlations may go up if there is a turmoil on the market!
- Alternative copulas: Student t copula, Clayton copula, Archimedean copula, Marshall-Olkin copula
- Random factor loadin $x_i = a(F)F + \sqrt{1 - a(F)}\, Z_i$
- Dynamic models – stocnastic moaeling or portfolio loss over time – structural (assets), reduced form (hazard rates), top down models (total loss)

Intensity of default stochastic modeling

- Necessary to model option-like credit derivatives and more complex products
- Structural stochastic models: stochastic asset value drives the event of default – unrealistic low PD for short maturities – can be solved introducing jumps or uncertain initial values



Source: Author

Reduced-form models

- Default intensity (hazard rate) treated as a stochastic variable
- Advantage: easier to calibrate, PDs and spreads observable
- Disadvantage: arrival of default not captured – introduction of doubly stochastic process where the arrival of default is a Poisson process conditional on the default intensity process, e.g.

$$d\lambda = a(b-\lambda)dt + \sigma dz$$

- Reduced form pricing: $P(t,T) = E\left[\exp\left(-\int_t^T (r(u)+\lambda(u))du\right) | t\right]$

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