Microeconomics 2

VŠE - Silvester van Koten

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    - At own risk – doesn't always work for some reason!

- Supportive materials:
  - **https://goo.gl/ahwNlk**

- The presentation is based on the following sources:

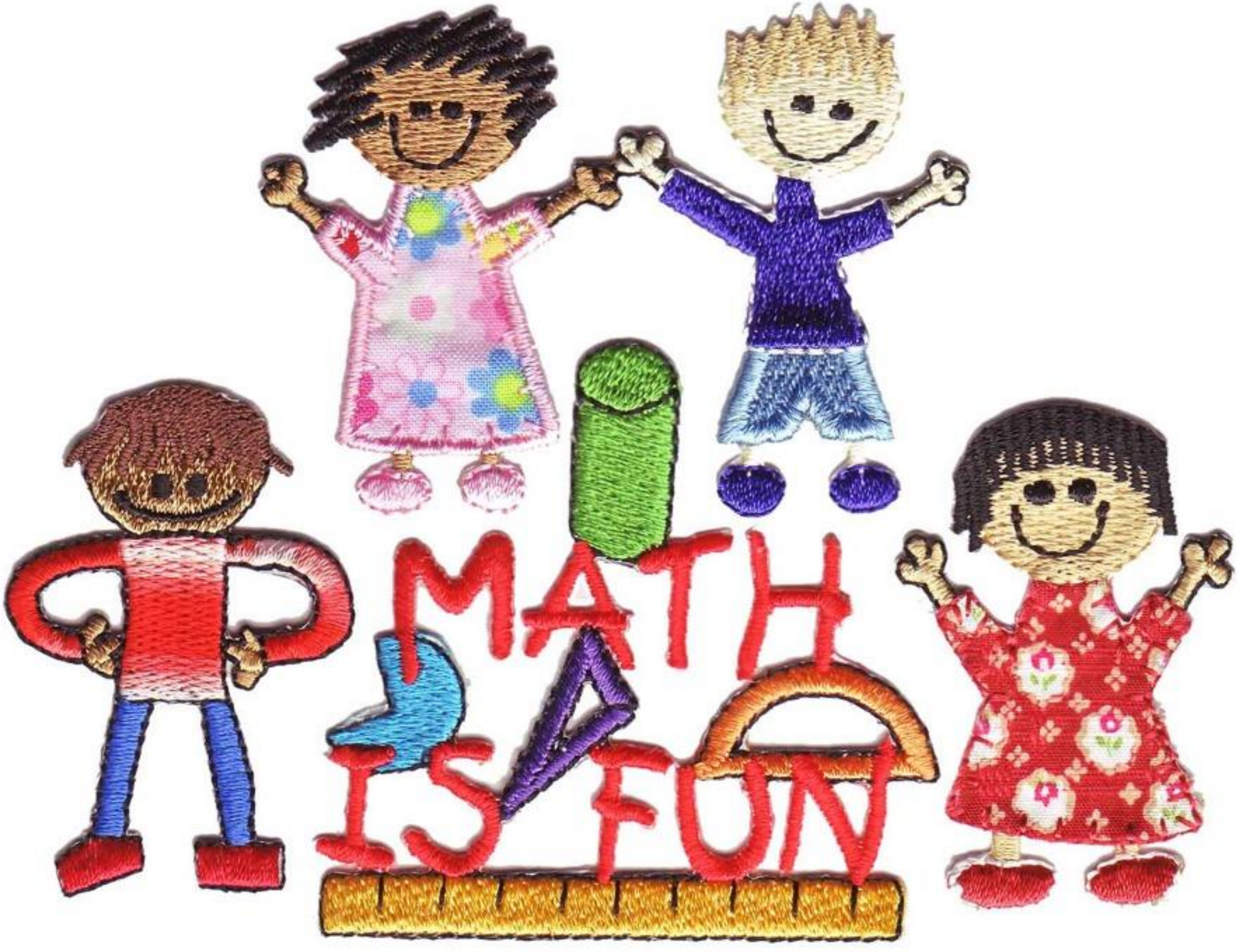
- Snyder Christopher M Walter Nicholson and Robert Stewart. 2015. Microeconomic Theory : Basic Principles and Extensions EMEA ed. Andover: Cengage Learning. +
- Varian Hal R. 2020. Intermediate Microeconomics : A Modern Approach Ninth ed. New York: W.W. Norton & Company. +
- Excerpts from Gravelle H O Hart and R Rees. 2004. Advanced Microeconomics.

Course contents:	
0. Mathematical basics 1) **Mathematical basics:**	N: Chap. 2 (p. 19-61) V: Math. Appendix
A. Consumer theory **2) The budget constraint:**	V: Chap 2
3) Consumer preferences:	N: Chap 2 V: Chap 3
4) Utility theory:	N: Chap 4 V: Chap 4
5) Choice:	N: Chap 4 V: Chap 5, 6, 7
6) Substitution and income effects:	N: Chap 5 V: Chap 8
7) "Duality": maximizing utility or minimizing expenditure	G: Chap 3 N: Chap 2
8) Consumer Surplus:	N: Chap 5 V: Chap 14
9) Decision making under risk: Chap 12	N: Chap 18 V: Chap 14

B. Theory of the Firm **10) Technology**: Chap 18	N: Chap 6 V: Chap 18
11) Profit maximizing: Chap 19	N: Chap 8 V: Chap 19
12) Cost minimizing: Chap 20, 21	N: Chap 7 V: Chap 20, 21
13) Supply: Chap 22, 23	N: Chap 9 V: Chap 22, 23
C. General equilibrium **14)** General equilibrium: Chap 31, 32	N: Chap 10 V: Chap 31, 32
15) Productive efficiency and exchange efficiency	N: Chap 10 V: Chap 31, 32

- Good news and bad news

Good news



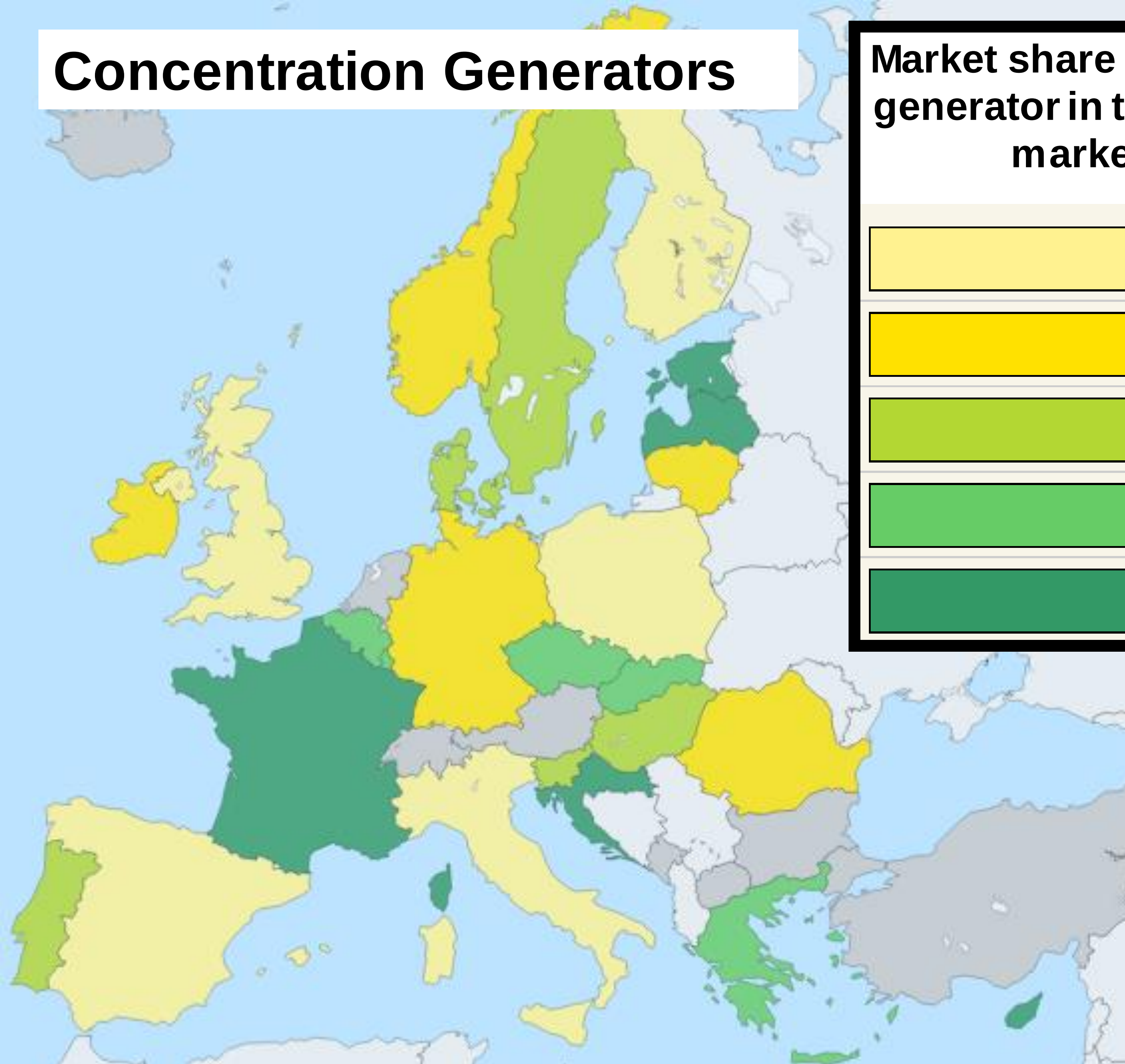
Some more good news

- Abstraction and math give a deep understanding

- What use is micro-economics for?

- **Academic/ consultancy/ policy report example**

Concentration Generators



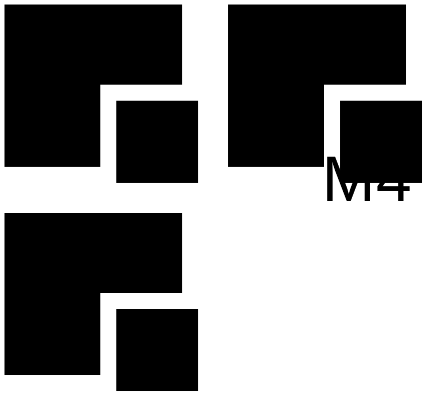
Market share of the largest generator in the electricity market in %

	17.4 - 28.0
	28.0 - 35.4
	35.4 - 56.3
	56.3 - 85.4
	85.4 - 100.0

M3

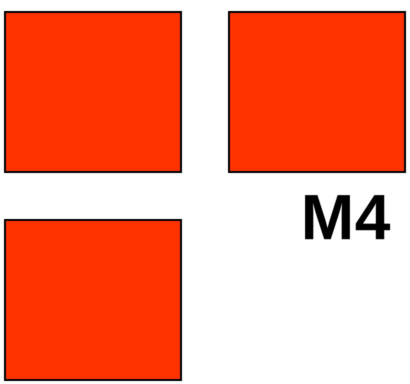
M4

M3 **M4**

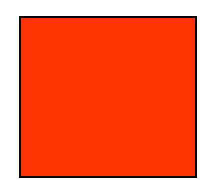


Divestment:
M4 has SAME aggregate
assets as M3

Brandts et al.



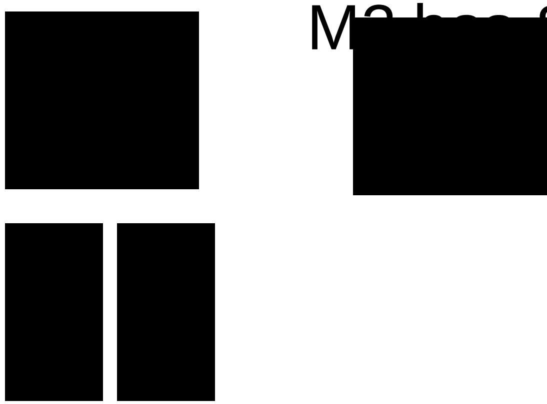
Entry:
M4 has MORE aggregate
assets than M3!



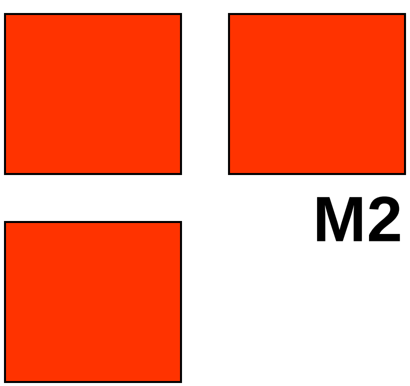
M3

M2

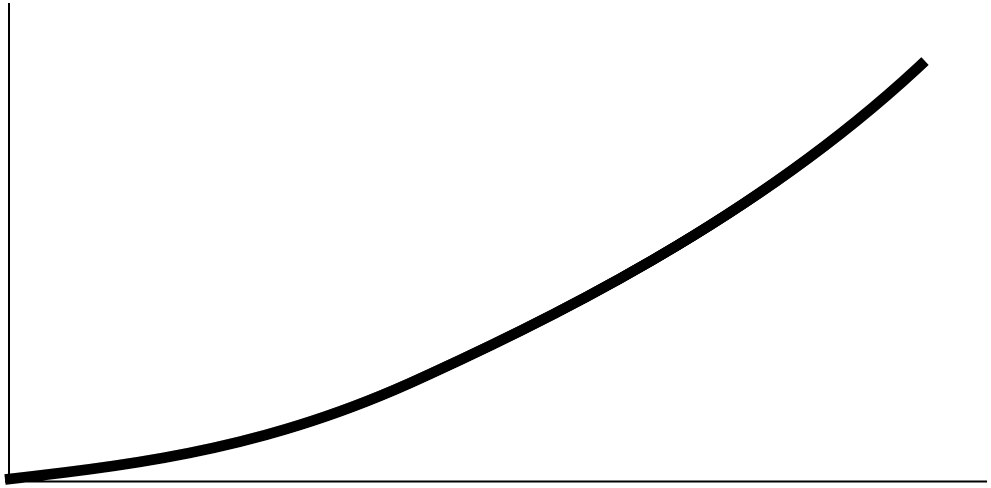
M3 **M2**



M[redacted] SAME aggregate
[redacted]sets as M3



Entry:
M2 has LESS aggregate assets than M3!

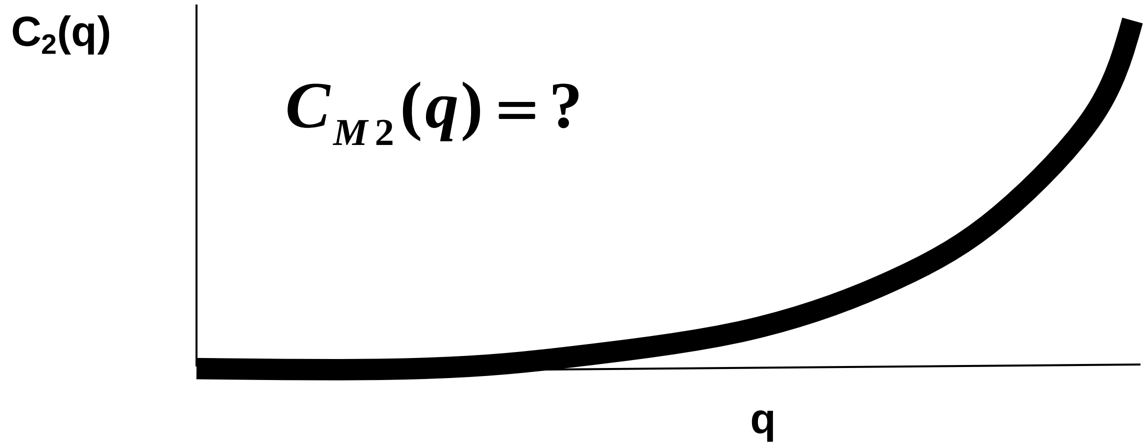


$$C_{M3}(q) = q^2$$

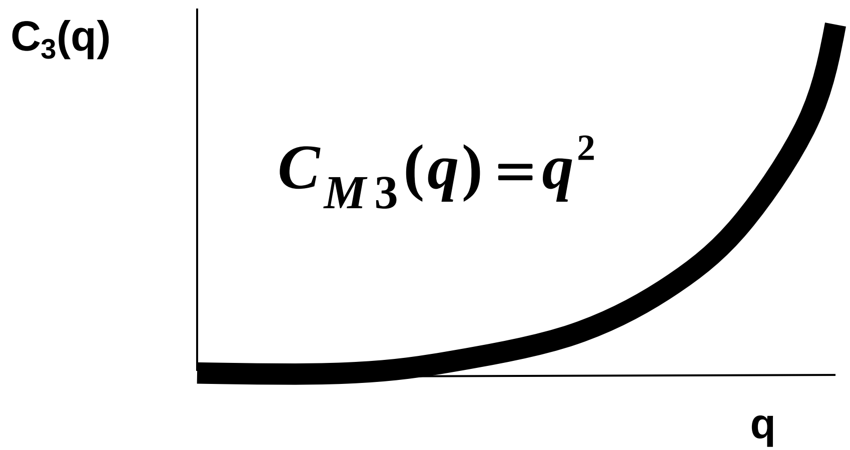
$$C'_{M3}(q) > 0 \textbf{ for } q > 0 \;\&\; C''_{M3}(q) > 0 \textbf{ for } q > 0$$

How should look M_{C2} and M_{C4}?

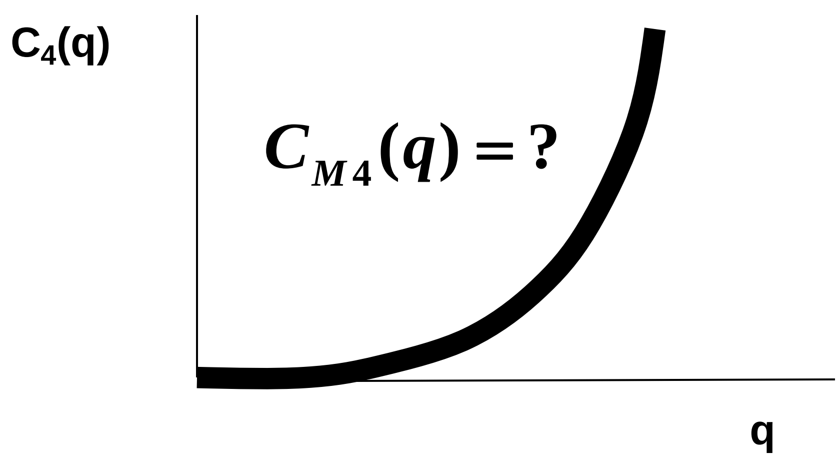
Cost function of a firm in a market with 2 producers



Cost function of a firm in a market with 3 producers



Cost function of a firm in a market with 4 producers



- How to find the new cost functions?

Market with THREE producers (original market)	
Total Production 3*q	**Total Costs 3*TC**
0	**0**
3	3
6	**12**
9	27
12	**48**

Market with TWO producers		Market with THREE producers (original market)	
Total Production 2*q	**Total Costs 2*TC**	Total Production 3*q	**Total Costs 3*TC**
0		0	**0**
		3	3
4			
6		6	**12**
8			
		9	27
12		12	**48**

Market with TWO producers		**Market with THREE producers (original market)**		**Market with FOUR producers**	
Total Production 2*q	**Total Costs 2*TC**	Total Production 3*q	**Total Costs 3*TC**	Total Production 4* q	**Total Costs 4*TC**
0	**0**	0	**0**	0	
		3	3		
4	**?**			4	
6	**12**	6	**12**	6	
8	**?**			8	
		9	27		
12	**48**	12	**48**	12	

Market with TWO producers		Market with THREE producers (original market)		Market with FOUR producers	
Total Production 2*q	**Total Costs 2*TC**	Total Production 3*q	**Total Costs 3*TC**	Total Production 4* q	**Total Costs 4*TC**
0	0	0	0	0	0
12	72	12	48	12	36

Adding competition by **Entry** (Brandts et al. 2008)

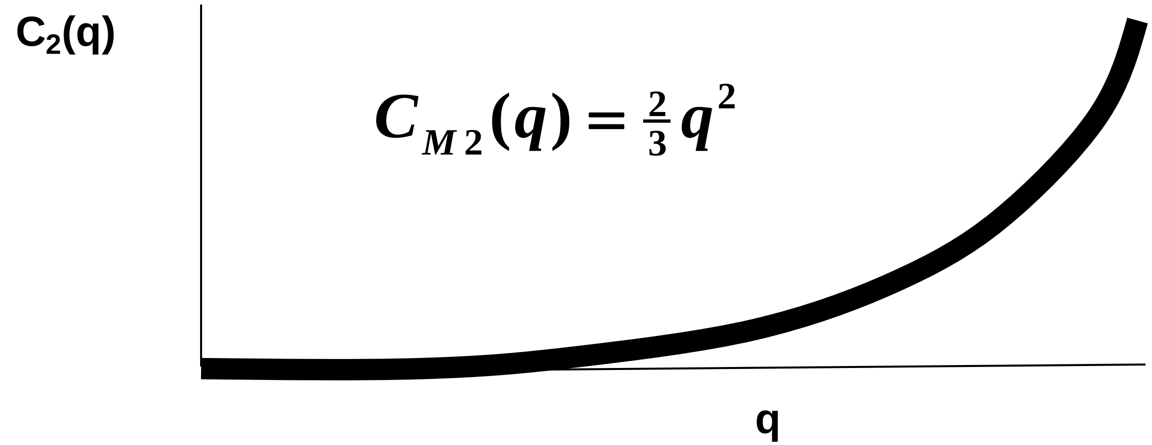
Entry: M2 has less assets and is thus more expensive

Entry: M4 has more assets and is thus cheaper

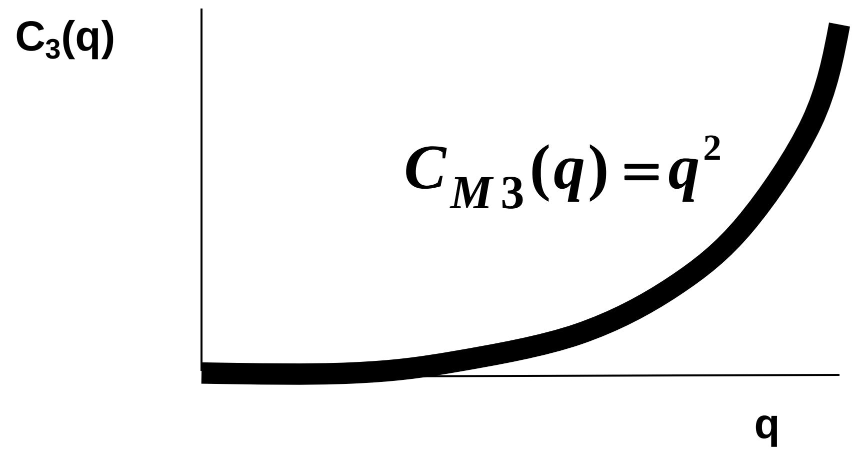
- How to find the new cost functions?
- For all values (not only multiplies of 12)

- Applying the basic theory of cost minimization makes finding the solution very easy

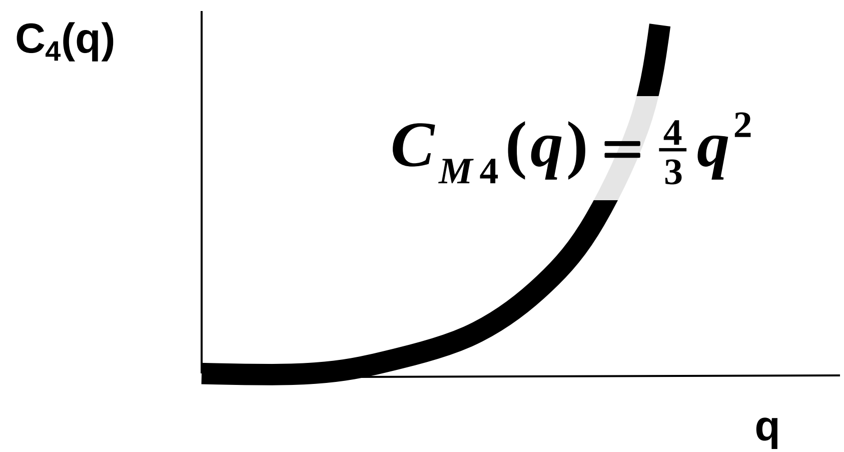
Cost function of a firm in a market with 2 producers



Cost function of a firm in a market with 3 producers



Cost function of a firm in a market with 4 producers



European Economic Review 64 (2013) 256–265

Contents lists available at ScienceDirect

European Economic Review

journal homepage: www.elsevier.com/locate/eer

Structural versus behavioral remedies in the deregulation of electricity markets: An experimental investigation motivated by policy concerns

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- While increasing the stock of production assets may have positive effects, the costs of creating these assets can be considerable.
- For example, building new power plants in the electricity industry is very costly.
- Introducing an equal-sized new competitor by entry in a market with 3 symmetrical competitors requires an increase in production assets by 33%.
- In a country such as the UK, an increase of that magnitude would correspond to an increase in electricity generation capacity of 27 GW and would cost – depending on whether the increase is realized by gas, coal or nuclear power plants – between 27 billion and 189 billion English pounds (Mott MacDonald, 2010, p. 58; Ofgem, 2013, p. 10).

- **Policy example**

TransAlta Renewables' Castle River project in Alberta

- **Alberta has set a “firm target” to obtain 30% of its electricity from renewable sources by 2030**
- **…** creating new, green jobs for Albertans
- To reach [this] target will require at least $7.95bn in fresh investment. That will translate into at least 7,200 new jobs.

- *Comment: if a policy creates jobs, is that a good thing?*

- Academics: for any research you need microeconomics
- Consultancy: you should be able to read and write reports that contain microeconomcis
- Policy: microeconomics to understand the +/- of different proposals
  - Eg: if a policy creates jobs, is that a good thing?
- Business: need a basic understanding of concepts of cost, profit, revenue.

- Mathematics:
  - Differentiation
  - Solving equations
  - Optimizing (maximizing & minimizing)
    - We are lucky here – often assuming convex problems

- **0. Mathematical basics: Varian Math. Appendix & Nicholson chap.2**

- **A. Consumer theory**

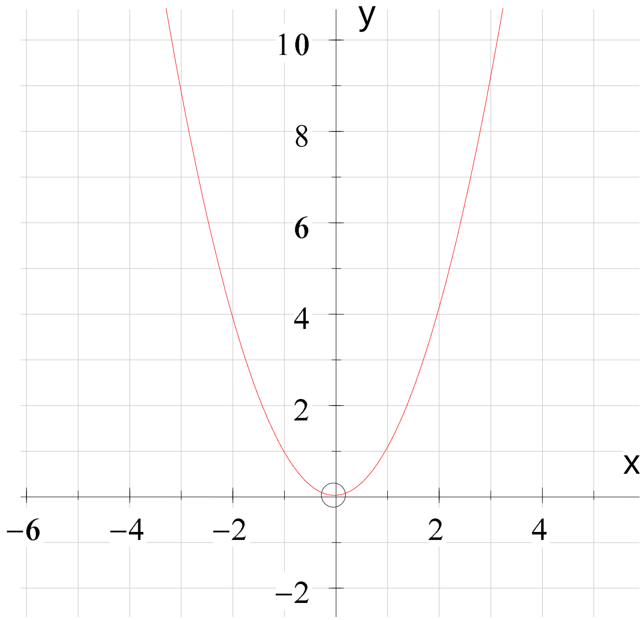
- **B. Theory of the Firm**

- **C. General equilibrium**

BASIC MATHEMATICS

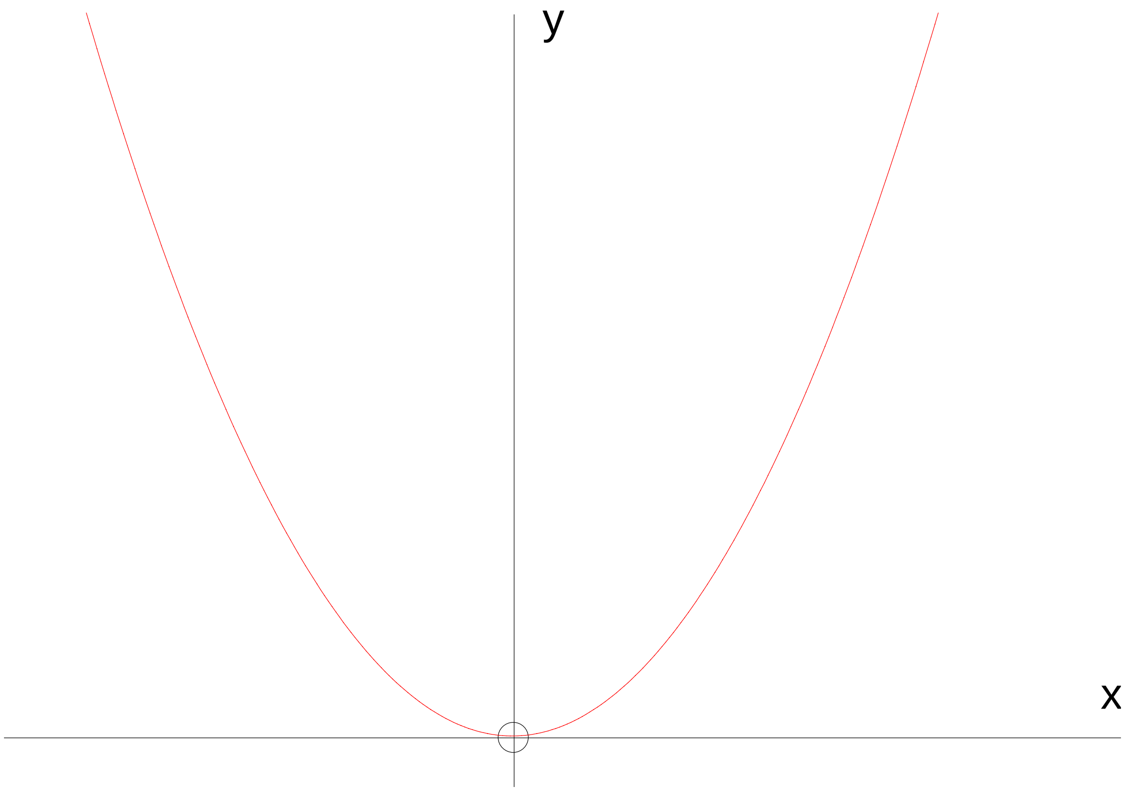
Differentiation

How can you find the gradient of a curve if it keeps changing??

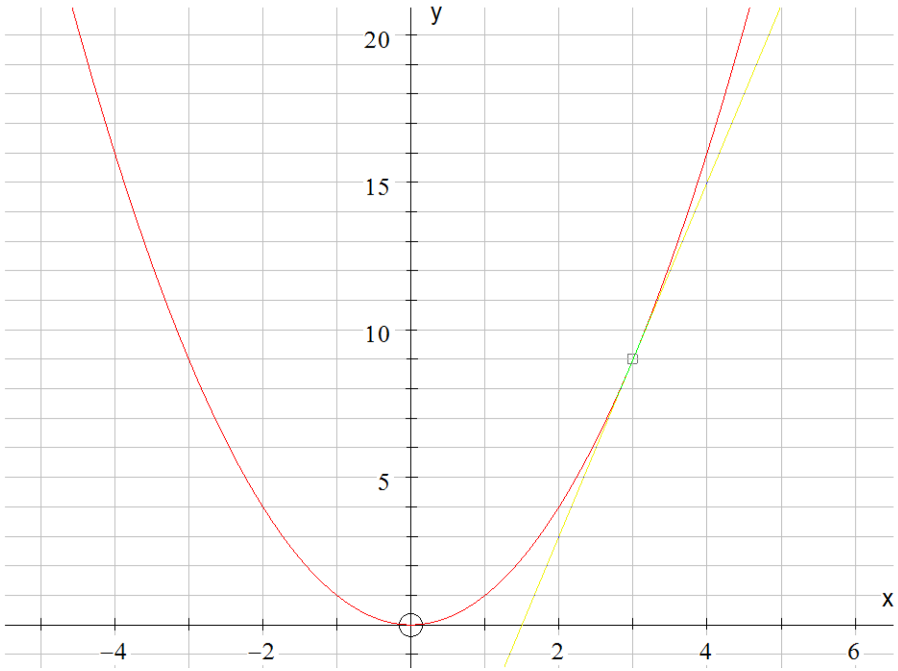


How can you find the gradient of a curve if it keeps changing??

E.g. the function $y = x^2$



The ideal way to find a gradient of a curve is to find the gradient of the tangent at the point we are interested in



Finding the gradient

- The process of finding the gradient of a curve is called **“differentiation”**

- You can **differentiate** any function to find its gradient

In general it can be shown that

$$f(x) = x^n$$

The function

$$f\,'(x) = nx^{n-1}$$

The **derivative** (or **differential**) of the function.
This is the **gradient function**

In other words

$f(x) = x^n$

$f'(x) = nx^{n-1}$

E.g. 1

$$f(x) = x^3$$

1) Multiply function by power of x

2) Subtract 1 from the power

-1

$$f'(x) = 3x^2$$

In other words

$f(x) = x^n$

$f'(x) = nx^{n-1}$

E.g. 1

$$f(x) = x^{-5}$$

1) Multiply function by power of x

2) Subtract 1 from the power

$$f'(x) = -5x^{-6}$$

Constant rule

$f(x) = x^n$

$f'(x) = nx^{n-1}$

$$f(x) = c$$

1) Such a function is just a horizontal line, so the slope is zero!

$$f'(x) = 0$$

Notation

$$y = f(x)$$

$$\frac{dy}{dx} = f'(x)$$

Find $\frac{dy}{dx}$ for the following curves.

$$y = 4x^3$$

Test differentiating polynomials

For any of the following questions there is exactly one correct answer.
Take a sheet of paper whenever you feel this might help.

The derivative of the function $x \to x^2$ is given by

$x \to x^2$	a
$x \to 2\,x^2$	b
$x \to 2\,x$	c

The derivative of the function $x \to x^3$ is given by

$x \to 2\,x^3$	a
$x \to 3\,x^2$	b
$x \to x^3/3$	c

The derivative of the function $x \to 4x^2 - 3x + 2$ is given by

$x \to 8x - 3$ **a**

$x \to 8x + 2$ **b**

$x \to 8x^2 - 3$ **c**

The derivative of the function $x \to -2x^2 + 6x - 4$ is given by

$x \to 4x + 6$ **a**

$x \to 4x - 4$ **b**

$x \to -4x + 6$ **c**

The derivative of the function $x \to 2x^3 - x^2 + 1$ is given by

$x \to 2x^2 - 2x$ **a**

$x \to 6x^2 - 2x$ **b**

$x \to 6x^2 - 2x + 1$ **c**

The derivative of the function $x \to -7x^3 + 2x^2 - x + 1$ is given by

$x \to -14x^2 + 4x - 1$ **a**

$x \to -21x^2 + 4x + 1$ **b**

$x \to -21x^2 + 4x - 1$ **c**

The derivative of the function $x \to (x^2 - 3)^2$ is given by

$x \to 4x^4 - 12x^2 + 6x - 9$ **a**

$x \to (2x - 3)^2$ **b**

$x \to 4x^3 - 12x$ **c**

$x \to 2(x^2 - 3)$ **d**

The graph of the derivative of a quadratic function is

a parabola **a**

a straight line **b**

a point **c**

The graph of the derivative of a linear function is

a parabola **a**

a straight line parallel to the x-axis **b**

a straight line parallel to the y-axis **c**

Test differentiating polynomials

For any of the following questions there is exactly one correct answer. Take a sheet of paper whenever you feel this might help.

The derivative of the function $x \to x^2$ is given by

$x \to x^2$	a
$x \to 2\,x^2$	b
<u>**$x \to 2\,x$**</u>	<u>**c**</u>

The derivative of the function $x \to x^3$ is given by

$x \to 2\,x^3$	a
<u>**$x \to 3\,x^2$**</u>	<u>**b**</u>
$x \to x^3/3$	c

The derivative of the function $x \to 4\,x^2 - 3\,x + 2$ is given by

<u>**$x \to 8\,x - 3$**</u>	<u>**a**</u>
$x \to 8\,x + 2$	b
$x \to 8\,x^2 - 3$	c

The derivative of the function $x \to -2\,x^2 + 6\,x - 4$ is given by

<u>**$x \to 4\,x + 6$**</u>	<u>**a**</u>
$x \to 4\,x - 4$	b
$x \to -4\,x + 6$	c

The derivative of the function $x \to 2\,x^3 - x^2 + 1$ is given by

$x \to 2\,x^2 - 2\,x$	a
<u>**$x \to 6\,x^2 - 2\,x$**</u>	<u>**b**</u>
$x \to 6\,x^2 - 2\,x + 1$	c

The derivative of the function $x \to -7\,x^3 + 2\,x^2 - x + 1$ is given by

$x \to -14\,x^2 + 4\,x - 1$	a
$x \to -21\,x^2 + 4\,x + 1$	b
<u>**$x \to -21\,x^2 + 4\,x - 1$**</u>	<u>**c**</u>

The derivative of the function $x \to (x^2 - 3)^2$ is given by

$x \to 4\,x^4 - 12\,x^2 + 6\,x - 9$	a
$x \to (2\,x - 3)^2$	b
<u>**$x \to 4\,x^3 - 12\,x$**</u>	<u>**c**</u>
$x \to 2\,(x^2 - 3)$	d

The graph of the derivative of a quadratic function is

a parabola	a
<u>**a straight line**</u>	<u>**b**</u>
a point	c

The graph of the derivative of a linear function is

a parabola	a
<u>**a straight line parallel to the x-axis**</u>	<u>**b**</u>
a straight line parallel to the y-axis	c

$$x^{y_1} \cdot x^{y_2} = x^{y_1+y_2}$$

$$x^{y_1} / x^{y_2} = x^{y_1} \cdot x^{-y_2} = x^{y_1-y_2}$$

$$\left(x^{y_1}\right)^{y_2} = x^{y_1 \cdot y_2}$$

We want a function that has:

$$f[x \cdot y] = f[x] + f[y]$$

We call it: $\ln[x]$ **We also define:** $\mathrm{e} = \ln^{-1}[1]$

$$\ln[x \cdot y] = \ln[x] + \ln[y]$$

$$\begin{aligned}\ln[x^y] &= \ln[\left(\mathrm{e}^{\ln[x]}\right)^y] = \ln[\mathrm{e}^{\ln[x]} \cdot \ldots \cdot \mathrm{e}^{\ln[x]}] = \\ &= \ln[\mathrm{e}^{\ln[x]}] + \ldots + \ln[\mathrm{e}^{\ln[x]}] \\ &= \mathrm{y} \cdot \ln[x]\end{aligned}$$

$$x^{y_1} \cdot x^{y_2} = x^{y_1+y_2}$$

$$x^{y_1} / x^{y_2} = x^{y_1} \cdot x^{-y_2} = x^{y_1-y_2}$$

$$\left(x^{y_1}\right)^{y_2} = x^{y_1 \cdot y_2}$$

We want a function that has:

$$f[x \cdot y] = f[x] + f[y]$$

I give a better setup and proofs next week

We call it: $\ln[x]$ **We also define:** $\mathrm{e} = \ln^{-1}[1]$

$$\ln[x \cdot y] = \ln[x] + \ln[y]$$

$$\ln[x^y] = \mathrm{y} \cdot \ln[x]$$

$$\ln[x / y] = \ln[x \cdot y^{-1}] = \ln[x] + \ln[y^{-1}] =$$

$$= \ln[x] - \ln[y]$$

A.9 Absolute Values and Logarithms

The **absolute value** of a number is a function $f(x)$ defined by the following rule:

$$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

Thus the absolute value of a number can be found by dropping the sign of the number. The absolute value function is usually written as $|x|$.

The (natural) **logarithm** or **log** of x describes a particular function of x, which we write as $y = \ln x$ or $y = \ln(x)$. The logarithm function is the unique function that has the properties

$$\ln(xy) = \ln(x) + \ln(y)$$

for all positive numbers x and y and

$$\ln(e) = 1.$$

(In this last equation, e is the base of natural logarithms which is equal to $2.7183\ldots$) In words, the log of the product of two numbers is the sum of the individual logs. This property implies another important property of logarithms:

$$\ln(x^y) = y\ln(x),$$

which says that the log of x raised to the power y is equal to y times the log of x.

A.10 Derivatives

The **derivative** of a function $y = f(x)$ is defined to be

$$\frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

In words, the derivative is the limit of the rate of change of y with respect to x as the change in x goes to zero. The derivative gives precise meaning to the phrase "the rate of change of y with respect to x for small changes in x." The derivative of $f(x)$ with respect to x is also denoted by $f'(x)$.

We have already seen that the rate of change of a linear function $y = ax + b$ is constant. Thus for this linear function

$$\frac{df(x)}{dx} = a.$$

For a nonlinear function the rate of change of y with respect to x will usually depend on x. We saw that in the case of $f(x) = x^2$, we had $\Delta y/\Delta x = 2x + \Delta x$. Applying the definition of the derivative

$$\frac{df(x)}{dx} = \lim_{\Delta x \to 0} 2x + \Delta x = 2x.$$

Thus the derivative of x^2 with respect to x is $2x$.

It can be shown by more advanced methods that if $y = \ln x$, then

$$\frac{df(x)}{dx} = \frac{1}{x}.$$

A.11 Second Derivatives

The **second derivative** of a function is the derivative of the derivative of that function. If $y = f(x)$, the second derivative of $f(x)$ with respect to x is written as $d^2 f(x)/dx^2$ or $f''(x)$. We know that

$$\frac{d(2x)}{dx} = 2$$
$$\frac{d(x^2)}{dx} = 2x.$$

Thus

$$\frac{d^2(2x)}{dx^2} = \frac{d(2)}{dx} = 0$$
$$\frac{d^2(x^2)}{dx^2} = \frac{d(2x)}{dx} = 2.$$

The second derivative measures the curvature of a function. A function with a negative second derivative at some point is concave near that point; its slope is decreasing. A function with a positive second derivative at a point is convex near that point; its slope is increasing. A function with a zero second derivative at a point is flat near that point.

Multiplication (product rule)

$$\frac{d\left(f[x]\cdot h[x]\right)}{dx} = h[x]\cdot\frac{df[x]}{dx} + f[x]\cdot\frac{dh[x]}{dx}$$

Composite functions (Chain rule)

$$\frac{df\left[h[x]\right]}{dx} = \frac{df\left[h[x]\right]}{dh}\cdot\frac{dh[x]}{dx}$$

Division = multiplication + chain rule

$$\frac{d\left(f[x]/h[x]\right)}{dx} = \frac{d\left(f[x]\cdot\left(h[x]\right)^{-1}\right)}{dx} = h[x]^{-1}\cdot\frac{df[x]}{dx} + f[x]\cdot\frac{dh[x]^{-1}}{dx}$$

$$= h[x]^{-1}\cdot\frac{df[x]}{dx} + f[x]\cdot-\left(h[x]\right)^{-2}\cdot\frac{dh[x]}{dx}$$

$$= \frac{h[x]\cdot\frac{df[x]}{dx} - f[x]\cdot\frac{dh[x]}{dx}}{\left(h[x]\right)^{2}}$$

A.12 The Product Rule and the Chain Rule

Suppose that $g(x)$ and $h(x)$ are both functions of x. We can define the function $f(x)$ that represents their product by $f(x) = g(x)h(x)$. Then the derivative of $f(x)$ is given by

$$\frac{df(x)}{dx} = g(x)\frac{dh(x)}{dx} + h(x)\frac{dg(x)}{dx}.$$

Given two functions $y = g(x)$ and $z = h(y)$, the **composite function** is

$$f(x) = h(g(x)).$$

For example, if $g(x) = x^2$ and $h(y) = 2y + 3$, then the composite function is

$$f(x) = 2x^2 + 3.$$

The **chain rule** says that the derivative of a composite function, $f(x)$, with respect to x is given by

$$\frac{df(x)}{dx} = \frac{dh(y)}{dy}\frac{dg(x)}{dx}.$$

In our example, $dh(y)/dy = 2$, and $dg(x)/dx = 2x$, so the chain rule says that $df(x)/dx = 2 \times 2x = 4x$. Direct calculation verifies that this is the derivative of the function $f(x) = 2x^2 + 3$.

Exercises

$$f(x) = \ln(x^2)$$

$$f'(x) = \frac{1}{x^2} \cdot 2x == \frac{2}{x}$$

$$f(x) = (x^2)^2$$

$$f'(x) = 2(x^2)^1 \cdot 2x = 4x^3$$

$$f(x) = \ln(1 + (x+1)^2)$$

$$f'(x) = \frac{1}{1 + (x+1)^2} 2(x+1)$$

$$f(x) = \ln(1 + (x^{-1}+1)^2)$$

$$f'(x) = \frac{1}{1 + (x^{-1}+1)^2} 2(x^{-1}+1) \cdot -1 \cdot \mathrm{x}^{-2}$$

- Partial derivates
- Total derivates

A.13 Partial Derivatives

Suppose that y depends on both x_1 and x_2, so that $y = f(x_1, x_2)$. Then the **partial derivative** of $f(x_1, x_2)$ with respect to x_1 is defined by

$$\frac{\partial f(x_1, x_2)}{\partial x_1} = \lim_{\Delta x_1 \to 0} \frac{f(x_1 + \Delta x_1, x_2) - f(x_1, x_2)}{\Delta x_1}.$$

The partial derivative of $f(x_1, x_2)$ with respect to x_1 is just the derivative of the function with respect to x_1, *holding* x_2 *fixed*. Similarly, the partial derivative with respect to x_2 is

$$\frac{\partial f(x_1, x_2)}{\partial x_2} = \lim_{\Delta x_2 \to 0} \frac{f(x_1, x_2 + \Delta x_2) - f(x_1, x_2)}{\Delta x_2}.$$

Partial derivatives have exactly the same properties as ordinary derivatives; only the name has been changed to protect the innocent (that is, people who haven't seen the ∂ symbol).

In particular, partial derivatives obey the chain rule, but with an extra twist. Suppose that x_1 and x_2 both depend on some variable t and that we define the function $g(t)$ by

$$g(t) = f(x_1(t), x_2(t)).$$

Then the derivative of $g(t)$ with respect to t is given by

$$\frac{dg(t)}{dt} = \frac{\partial f(x_1, x_2)}{\partial x_1} \frac{dx_1(t)}{dt} + \frac{\partial f(x_1, x_2)}{\partial x_2} \frac{dx_2(t)}{dt}.$$

When t changes, it affects both $x_1(t)$ and $x_2(t)$. Therefore, we need to calculate the derivative of $f(x_1, x_2)$ with respect to each of those changes.

Exercises

$$f[x,y] = x + \ln[\mathrm{y}]$$

$$\frac{\partial f[x,y]}{\partial x} = 1$$

$$\frac{\partial f[x,y]}{\partial y} = \frac{1}{y}$$

$$f[x,y,z] = z^2 xy + \ln[y-x]$$

$$\frac{\partial f[x,y,z]}{\partial x} = z^2 y + \frac{1}{y-x} \cdot -1$$

$$\frac{\partial f[x,y,z]}{\partial y} = z^2 x + \frac{1}{y-x}$$

$$\frac{\partial f[x,y,z]}{\partial z} = 2zxy$$

Exercises

$$f[x, y] = x + y^2$$

$$\frac{\partial f[x, y]}{\partial x} = 1$$

$$\frac{\partial f[x, y]}{\partial y} = 2y$$

$$y = g(x)$$

$$f(x, g(x)) = x + g(x)^2$$

$$\frac{df(x, \mathrm{g(x)})}{dx} = 1 + 2g(x) \cdot g'(x)$$

Exercises

$$y = g(x) = \ln[x]$$

$$\frac{\partial f(x, g(x))}{\partial x} = 1 + 2g(x) \cdot g'(x)$$

$$= 1 + 2\ln[x] \cdot \frac{1}{x}$$

$$= 1 + \frac{2\ln[x]}{x}$$

$$f(x, y) = f(x, \ln[x])$$

$$= x + \ln[x]^2$$

$$\frac{\partial f(x, y)}{\partial x} = 1 + 2\ln[x] \cdot \frac{1}{x}$$

A.14 Optimization

If $y = f(x)$, then $f(x)$ achieves a **maximum** at x^* if $f(x^*) \geq f(x)$ for all x. It can be shown that if $f(x)$ is a smooth function that achieves its maximum value at x^*, then

$$\frac{df(x^*)}{dx} = 0$$
$$\frac{d^2 f(x^*)}{dx^2} \leq 0.$$

These expressions are referred to as the **first-order condition** and the **second-order condition** for a maximum. The first-order condition says that the function is flat at x^*, while the second-order condition says that the function is concave near x^*. Clearly both of these properties have to hold if x^* is indeed a maximum.

We say that $f(x)$ achieves its **minimum** value at x^* if $f(x^*) \leq f(x)$ for all x. If $f(x)$ is a smooth function that achieves its minimum at x^*, then

$$\frac{df(x^*)}{dx} = 0$$
$$\frac{d^2 f(x^*)}{dx^2} \geq 0.$$

The first-order condition again says that the function is flat at x^*, while the second-order condition now says that the function is convex near x^*.

If $y = f(x_1, x_2)$ is a smooth function that achieves its maximum or minimum at some point (x_1^*, x_2^*), then we must satisfy

$$\frac{\partial f(x_1^*, x_2^*)}{\partial x_1} = 0$$
$$\frac{\partial f(x_1^*, x_2^*)}{\partial x_2} = 0.$$

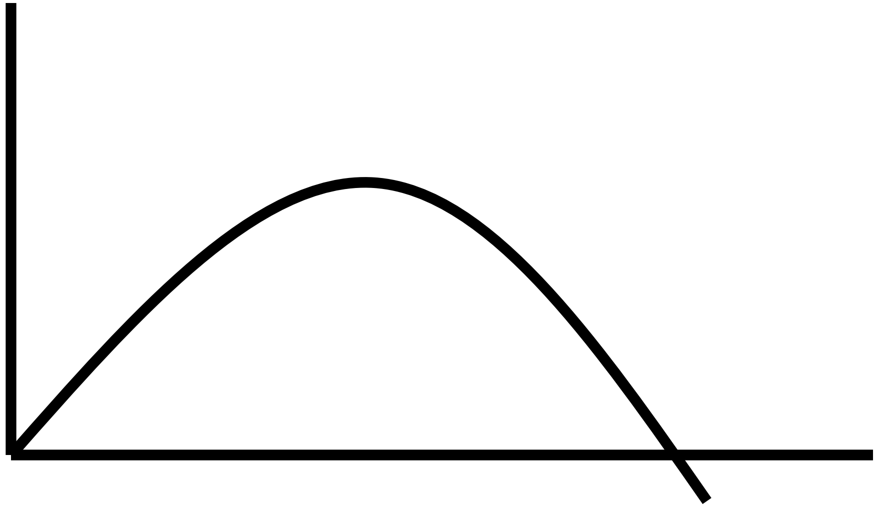
These are referred to as the **first-order conditions**. There are also second-order conditions for this problem, but they are more difficult to describe.

Exercises

$$Max\ f(x) = (1-x)x$$

$$0 = \frac{df(x)}{dx} = -1 \cdot x + (1-x) = 1 - 2x \leftrightarrow x = 1/2$$

$$\frac{d^2 f(x)}{dx^2} = -2$$



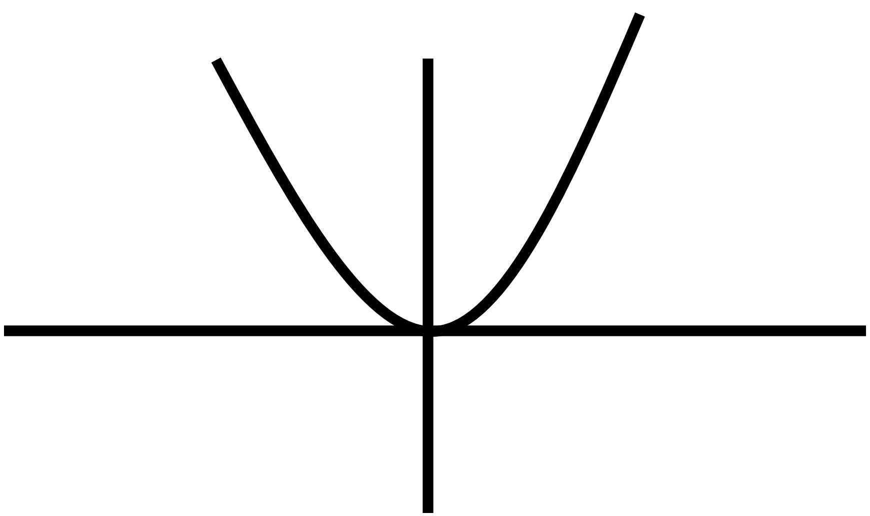
Exercises

$$Max\ f(x) = x^2$$

$$0 = \frac{df(x)}{dx} = 2x \leftrightarrow x = 0$$

$$\frac{d^2 f(x)}{dx^2} = 2 > 0$$

Not a maximum, but a minimum!



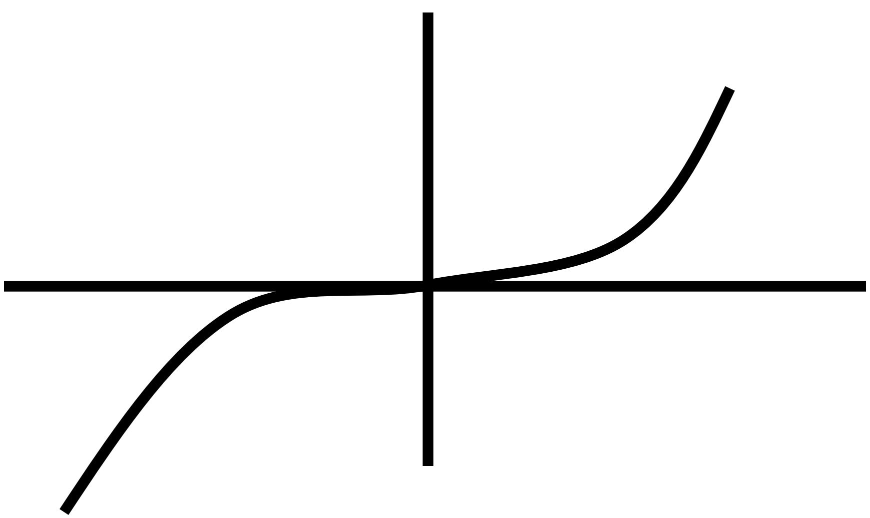
Exercises

$$Max\ f(x) = x^3$$

$$0 = \frac{df(x)}{dx} = 3x^2 \leftrightarrow x = 0$$

$$\frac{d^2 f(x)}{dx^2} = 6x = 0\,?$$

Not a maximum!



INTERMEDIATE MATHEMATICS

A.15 Constrained Optimization

Often we want to consider the maximum or minimum of some function over some restricted values of (x_1, x_2). The notation

$$\max_{x_1, x_2} f(x_1, x_2)$$

$$\text{such that } g(x_1, x_2) = c.$$

means

find x_1^* and x_2^* such that $f(x_1^*, x_2^*) \geq f(x_1, x_2)$ for all values of x_1 and x_2 that satisfy the equation $g(x_1, x_2) = c$.

The function $f(x_1, x_2)$ is called the **objective function**, and the equation $g(x_1, x_2) = c$ is called the **constraint**. Methods for solving this kind of constrained maximization problem are described in the Appendix to Chapter 5.

- **Lagrange**

$$y = f[x_1, x_2] \qquad g[x_1, x_2] = 0$$

$$\max_{x_1, x_2} f[x_1, x_2], \quad s.t. \quad g[x_1, x_2] = 0$$

$$L = f[x_1, x_2] + \lambda \cdot g[x_1, x_2]$$

$$FOC:$$

$$L_1 = 0$$

$$L_2 = 0$$

$$L_\lambda = 0$$

What if we have more restrictions?

$$y = f[x_1, x_2] \qquad \forall i : g_i[x_1, x_2] = 0$$

$$\max_{x_1, x_2} f[x_1, x_2], \quad s.t. \ \forall i : g_i[x_1, x_2] = 0$$

$$L = f[x_1, x_2] + \sum_{i=1}^{n} \lambda_i \cdot g_i[x_1, x_2]$$

$$FOC:$$

$$L_1 = 0$$

$$L_2 = 0$$

$$L_{\lambda_1} = L_{\lambda_{\dots}} = L_{\lambda_n} = 0$$

Are we sure that the restriction binds?

Otherwise we need: Karush-Kuhn-Tucker conditions (KKT conditions)

- Examples

$$Max\ y = x_1 + x_2 \qquad g_1[x_1, x_2] = 15 - x_1$$

$$g_2[x_1, x_2] = 25 - x_2$$

$$L = x_1 + x_2 + \lambda_1 \cdot (15 - x_1) + \lambda_2 \cdot (25 - x_2)$$

$FOC:$

$$0 = L_1 = 1 - \lambda_1$$

$$0 = L_2 = 1 - \lambda_2$$

$$\left.\begin{aligned} 0 &= L_{\lambda_1} = 15 - x_1 \\ 0 &= L_{\lambda_2} = 25 - x_2 \end{aligned}\right\} \qquad \begin{aligned} x_1 &= 15 \\ x_2 &= 25 \end{aligned}$$

$$\text{MAX}: y = x_1 \cdot x_2$$

$$g_1[x_1, x_2] = 100 - (x_1 + x_2)$$

$$L = x_1 \cdot x_2 + \lambda_1 \cdot (100 - (x_1 + x_2))$$

$$FOC:$$

$$\left.\begin{aligned} 0 &= L_1 = x_2 - \lambda_1 \\ 0 &= L_2 = x_1 - \lambda_1 \end{aligned}\right\} \quad x_2 = \lambda = x_1$$

$$\underbrace{0 = L_{\lambda_1} = 100 - (x_1 + x_2)}$$

$$2x_1 - 100 = 0 \qquad x_1 = x_2 = 50$$

- Karash-Kuhn-Tucker

- Same problem, but we are just not completely sure g[x] will bind

$$\max_{x_1, x_2} \ f[x_1, x_2], \quad s.t. \quad g[x_1, x_2] \le 0$$

$$L = f[x_1, x_2] + \lambda \cdot g[x_1, x_2]$$

$FOC:$

$$L_1 = 0$$
$$L_2 = 0$$

$$\lambda \cdot L_\lambda = 0, \lambda \ge 0, L_\lambda \le 0$$

- Example

$$\mathrm{MAX}: y = x_1 \cdot x_2$$

$$g_1[x_1, x_2] = (x_1 + x_2) - 100 \leq 0$$

$$L = x_1 \cdot x_2 + \lambda_1 \cdot \left((x_1 + x_2) - 100\right)$$

$$FOC:$$

$$\left.\begin{aligned} 0 &= L_1 = \left(x_2 - \lambda_1\right) \\ 0 &= L_2 = \left(x_1 - \lambda_1\right) \end{aligned}\right\} \quad x_2 = \lambda_1 = x_1$$

$$\underbrace{0 = \lambda_1 \cdot L_{\lambda_1} = \lambda_1 \cdot \left(\left(x_1 + x_2\right)}_{} - 100\right)$$

$$\text{if } \lambda_1 = 0 \quad \Rightarrow x_2 = x_1 = \lambda_1 = 0$$

not the maximum

$$\text{MAX}: y = x_1 \cdot x_2$$

$$g_1[x_1, x_2] = (x_1 + x_2) - 100 \leq 0$$

$$L = x_1 \cdot x_2 + \lambda_1 \cdot \left((x_1 + x_2) - 100\right)$$

$$FOC:$$

$$\left.\begin{aligned} 0 &= L_1 = \left(x_2 - \lambda_1\right) \\ 0 &= L_2 = \left(x_1 - \lambda_1\right) \end{aligned}\right\} \quad x_2 = \lambda_1 = x_1$$

$$0 = \lambda_1 \cdot L_{\lambda_1} = \lambda_1 \cdot \left((x_1 + x_2) - 100\right)$$

$$\text{if } \lambda_1 > 0 \quad \Rightarrow x_2 = x_1 > 0$$

$$2x_1 - 100 = 0 \qquad x_1 = x_2 = 50$$

- Lets now look at non-negativity constraints in a slightly different problem

- Suppose we know that g[x] binds
- We just are worried that our x1, x2 might take negative values!

$$\max_{x_1, x_2} \; f[x_1, x_2], \quad s.t. \quad g[x_1, x_2] = 0$$

$$x_1 \geq 0, x_2 \geq 0$$

$$L = f[x_1, x_2] + \lambda \cdot g[x_1, x_2]$$

$$FOC:$$

$$x_1 \cdot L_1 = 0, x_1 \geq 0, L_1 \geq 0$$

$$x_2 \cdot L_2 = 0, x_2 \geq 0, L_2 \geq 0$$

$$L_\lambda = 0$$

$$\max_{x_1, x_2} \; f[x_1, x_2], \quad s.t. \quad g[x_1, x_2] = 0$$

$$x_1 \geq 0, x_2 \geq 0$$

$$L = f[x_1, x_2] + \lambda \cdot g[x_1, x_2]$$

$$FOC:$$

$$x_1 \cdot L_1 = 0, x_1 \geq 0, L_1 \leq 0$$

$$x_2 \cdot L_2 = 0, x_2 \geq 0, L_2 \leq 0$$

$$L_\lambda = 0$$

I corrected an error! (I wrote)

$$L_1 \geq 0, L_2 \geq 0$$

- Example

$$\text{MIN}: y = x_1 \cdot x_2$$

$$g_1[x_1, x_2] = (x_1 + x_2) - 100 = 0$$

$$x_1 \geq 0, x_2 \geq 0$$

$$L = x_1 \cdot x_2 + \lambda_1 \cdot \left((x_1 + x_2) - 100\right)$$

$$FOC:$$

$$\left.\begin{array}{l} 0 = x_1 \cdot L_1 = x_1 \cdot \left(x_2 - \lambda_1\right) \\ 0 = x_2 \cdot L_2 = x_2 \cdot \left(x_1 - \lambda_1\right) \end{array}\right\} \boxed{\begin{array}{l} \text{if } x_1 > 0 \;\&\; x_2 > 0 \\ \quad x_2 = \lambda_1 = x_1 \end{array}}$$

$$\underbrace{0 = L_{\lambda_1} = (x_1 + x_2) - 100}$$

$$2x_1 - 100 = 0 \qquad x_1 = x_2 = 50 \qquad \boxed{y = 2500}$$

- This maximizes the function!
- We need here to check the 2nd order condition

$$L_2 = \frac{\partial\left(x_2 - \lambda_1\right)}{\partial x_2} = 1 > 0$$

$$L_1 = \frac{\partial\left(x_1 - \lambda_1\right)}{\partial x_1} = 1 > 0$$

- Yep, the solution we found is the maximum

$$\text{MIN}: y = x_1 \cdot x_2$$
$$g_1[x_1, x_2] = (x_1 + x_2) - 100 = 0$$
$$x_1 \geq 0, x_2 \geq 0$$

$$L = x_1 \cdot x_2 + \lambda_1 \cdot \left((x_1 + x_2) - 100\right)$$

$$FOC:$$

$$\left.\begin{aligned} 0 &= x_1 \cdot L_1 = x_1 \cdot \left(x_2 - \lambda_1\right) \\ 0 &= x_2 \cdot L_2 = x_2 \cdot \left(x_1 - \lambda_1\right) \end{aligned}\right\} \boxed{\begin{aligned} &\text{if } x_1 = 0 \;\&\; x_2 > 0 \\ &x_2 - \lambda_1 > 0 \\ &x_1 = \lambda_1 \;\Rightarrow \lambda_1 = 0 \end{aligned}}$$

$$\underbrace{0 = L_{\lambda_1} = x_1 + x_2 - 100}$$

$$x_2 - 100 = 0 \quad x_1 = 0 \;\&\; x_2 = 100 \quad \boxed{y = 0}$$

$$\mathrm{MIN}: y = x_1 \cdot x_2$$

$$g_1[x_1, x_2] = (x_1 + x_2) - 100 = 0$$

$$x_1 \geq 0, x_2 \geq 0$$

$$L = x_1 \cdot x_2 + \lambda_1 \cdot \left((x_1 + x_2) - 100\right)$$

$$FOC:$$

$$\left.\begin{aligned} 0 &= x_1 \cdot L_1 = x_1 \cdot \left(x_2 - \lambda_1\right) \\ 0 &= x_2 \cdot L_2 = x_2 \cdot \left(x_1 - \lambda_1\right) \end{aligned}\right\} \boxed{\begin{array}{l} \text{if } x_2 = 0 \;\&\; x_1 > 0 \\ x_2 = \lambda_1 \Rightarrow \lambda_1 = 0 \\ x_1 - \lambda_1 > 0 \end{array}}$$

$$\underbrace{0 = L_{\lambda_1} = x_1 + x_2 - 100}$$

$$x_1 - 100 = 0 \qquad x_1 = 100 \;\&\; x_2 = 0 \qquad \boxed{y = 0}$$

- Example minimizing costs

$$\boxed{c = w_1 \cdot x_1 + w_2 \cdot x_2} \qquad \boxed{g(x_1, x_2) = x_1 + x_2 = y}$$

$$L = w_1 \cdot x_1 + w_2 \cdot x_2 + \lambda_1 \cdot (x_1 + x_2 - y)$$

$$FOC:$$

$$0 = x_1 \cdot L_1 = x_1 \cdot \left(w_1 + \lambda_1\right) \quad x_1 \geq 0, L_1 \geq 0$$

$$0 = x_2 \cdot L_2 = x_2 \cdot \left(w_2 + \lambda_1\right) \quad x_2 \geq 0, L_2 \geq 0$$

$$0 = L_{\lambda_1} = x_1 + x_2 - y$$

$$0 = x_1 \cdot L_1 = x_1 \cdot (w_1 + \lambda_1) \qquad 0 = L_{\lambda_1} = x_1 + x_2 - y$$
$$0 = x_2 \cdot L_2 = x_2 \cdot (w_2 + \lambda_1)$$

$$\text{case 1: } x_1 > 0 \; \& \; x_2 > 0$$
$$(w_2 + \lambda_1) = (w_1 + \lambda_1) = 0 \qquad \Leftrightarrow w_2 = w_1$$

$$\text{case 2: } x_1 = 0 \; \& \; x_2 > 0$$
$$(w_1 + \lambda_1) > 0$$
$$\Leftrightarrow w_1 > w_2$$
$$(w_2 + \lambda_1) = 0$$

$$\text{case 3: } x_1 > 0 \; \& \; x_2 = 0$$
$$(w_2 + \lambda_1) > 0$$
$$\Leftrightarrow w_2 > w_1$$
$$(w_1 + \lambda_1) = 0$$

$$0 = x_1 \cdot L_1 = x_1 \cdot \left(w_1 + \lambda_1 \right)$$
$$0 = x_2 \cdot L_2 = x_2 \cdot \left(w_2 + \lambda_1 \right)$$
$$0 = L_{\lambda_1} = x_1 + x_2 - y$$

case 4: $x_1 = 0 \;\&\; x_2 = 0$

- Read N (Nicholson, Microeconomic Theory), p. 20-29, 38-46

- We will use the above methods *many* times

- Some more wild parts: envelope theorem (we leave this for later)

- **A. Consumer theory**
- **2. The budget constraint:** Chap 2

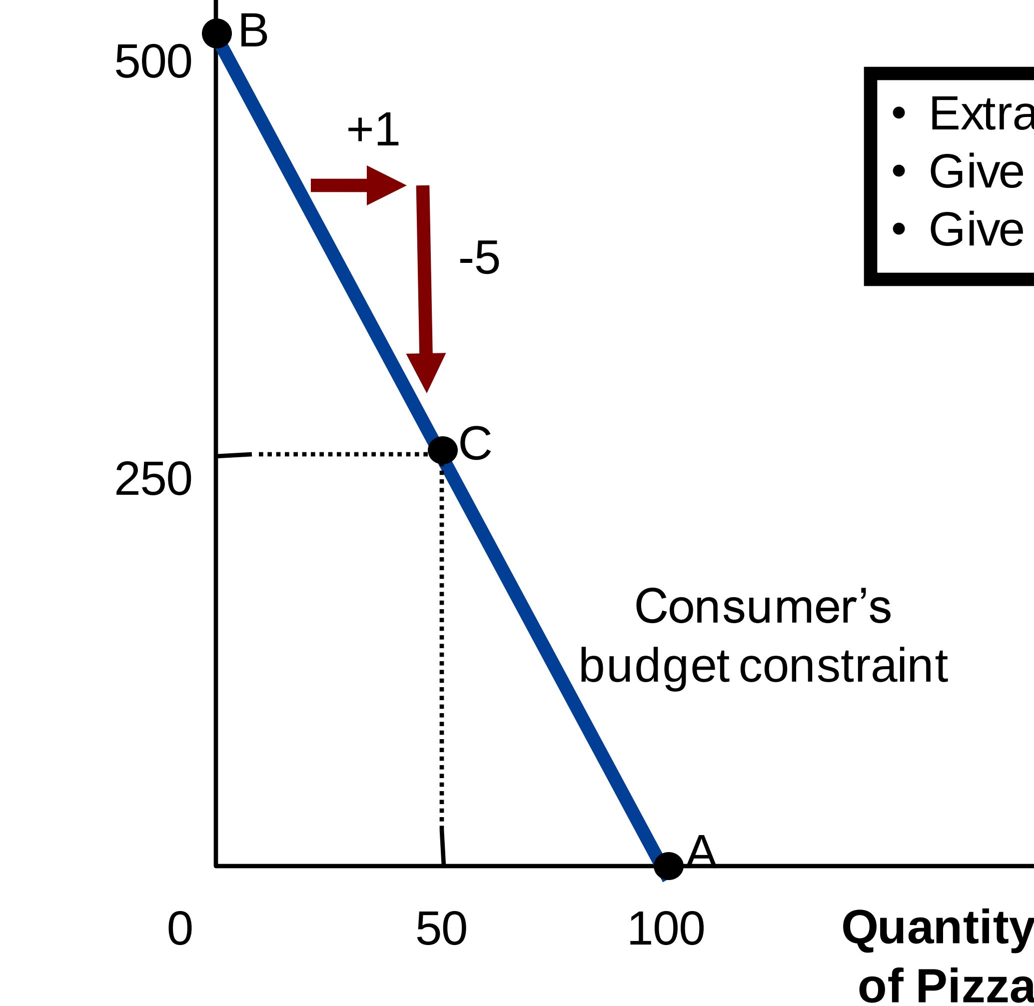
The Consumer's Budget Constraint

Budget	Cost of Cola	Cost of Pizza
1000	$2	$10

Pints of Cola`	Number of Pizzas	Spending on Pepsi	Spending on Pizza	Total Spending
0	100	$ 0	$1,000	$1,000
50	90	100	900	1,000
100	80	200	800	1,000
150	70	300	700	1,000
200	60	400	600	1,000
250	50	500	500	1,000
300	40	600	400	1,000
350	30	700	300	1,000
400	20	800	200	1,000
450	10	900	100	1,000
500	0	1,000	0	1,000

The Consumer’s Budget Constraint

Budget	Price of Cola	Price of Pizza
1000	$2	$10



- Extra cost of $10
- Give up cola for $10
- Give up 5 units of cola!

Slope = -5

Slope gives the units of cola you need to give up for 1 more pizza

Slope of budget constraint is called:

“the Marginal Rate of Transformation” (**MRT**)

- $Q_C*2 + Q_P*10=1000$
- $Q_C*2 =1000- Q_P*10$
- $Q_C = (1000- Q_P*10)/2$
- $Q_C = 1000/2 – (10/2)* Q_P$
- $Q_C = 500 – 5* Q_P$

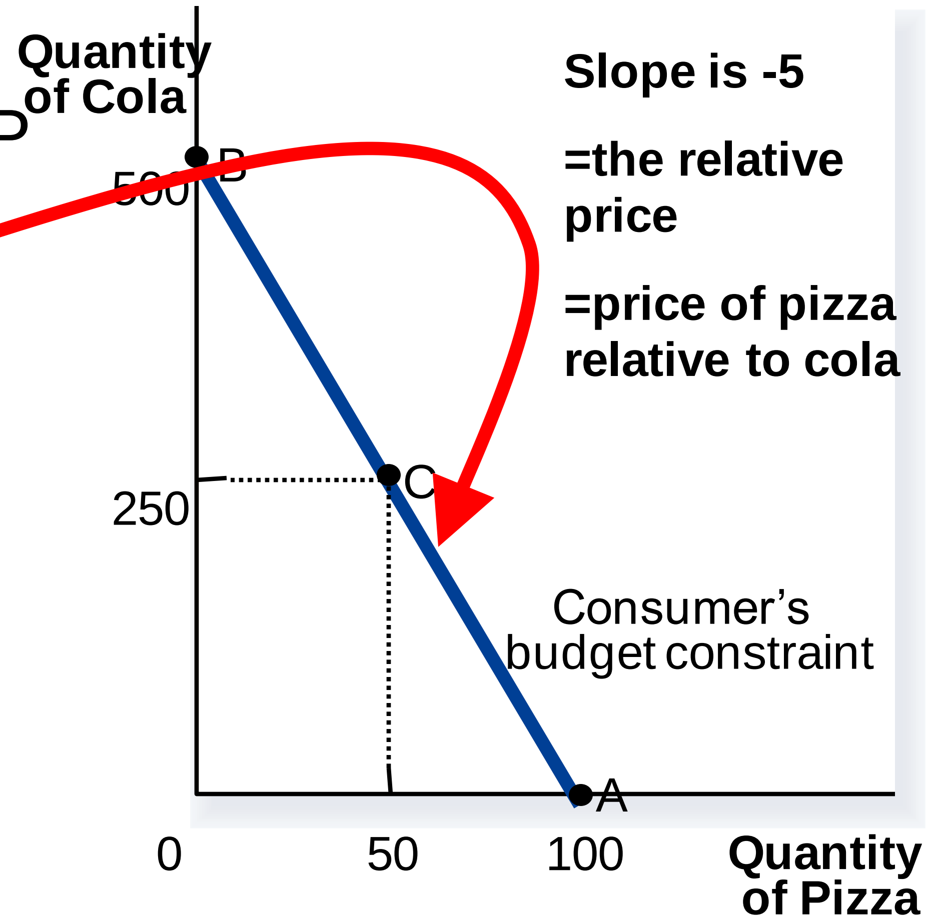
Budget	Cost of Cola	Cost of Pizza
1000	\$2	\$10

M=1000	P_C=2	P_P=10

$$Q_C = M/P_C - (P_P/P_C)^* \, Q_P$$

$$Q_C = 1000/2 - (10/2)^* \, Q_P$$

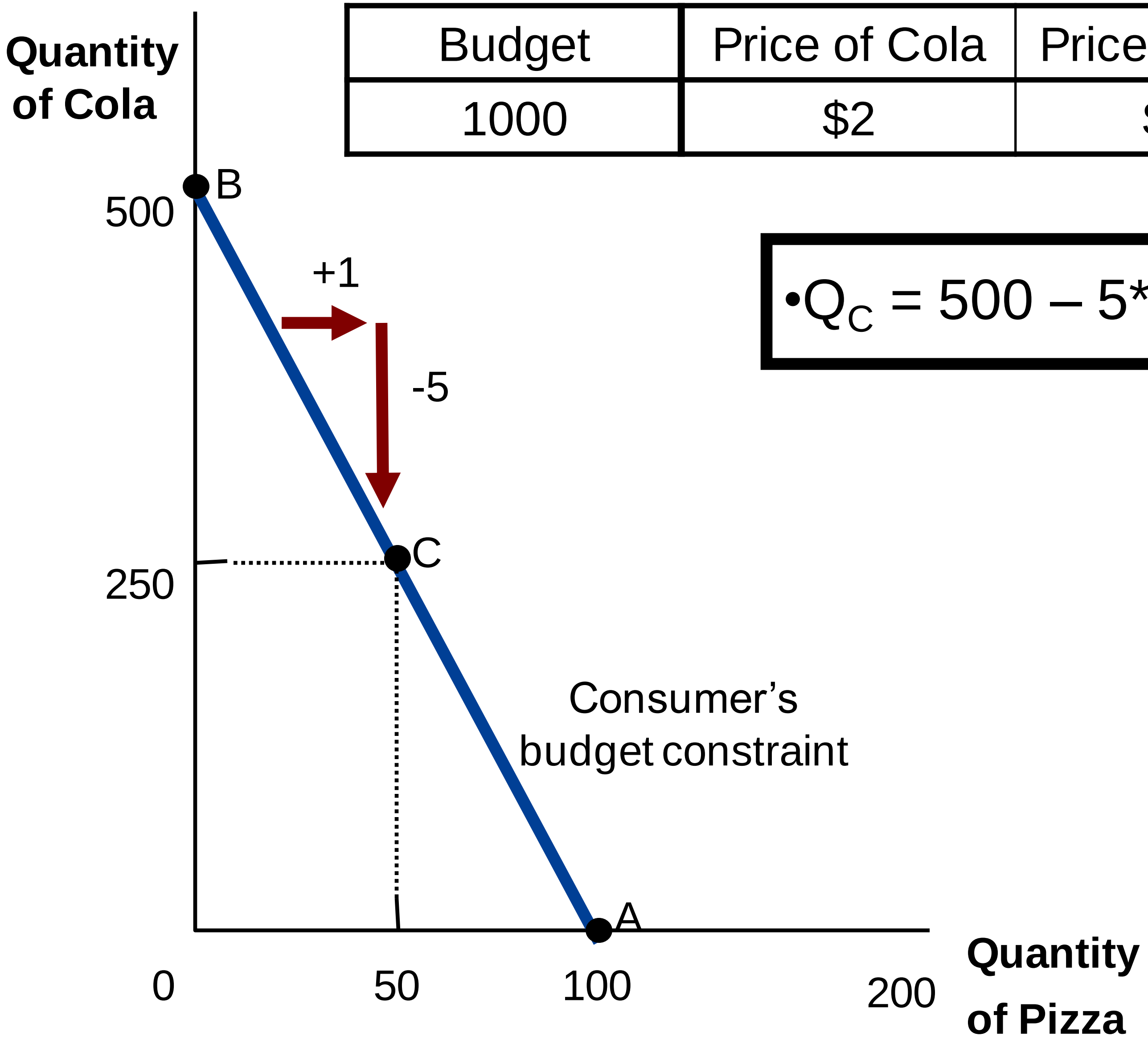
$$Q_C = 500 - 5 \, Q_P$$



The Consumer's Budget Constraint

Budget	Price of Cola	Price of Pizza
1000	$2	$10

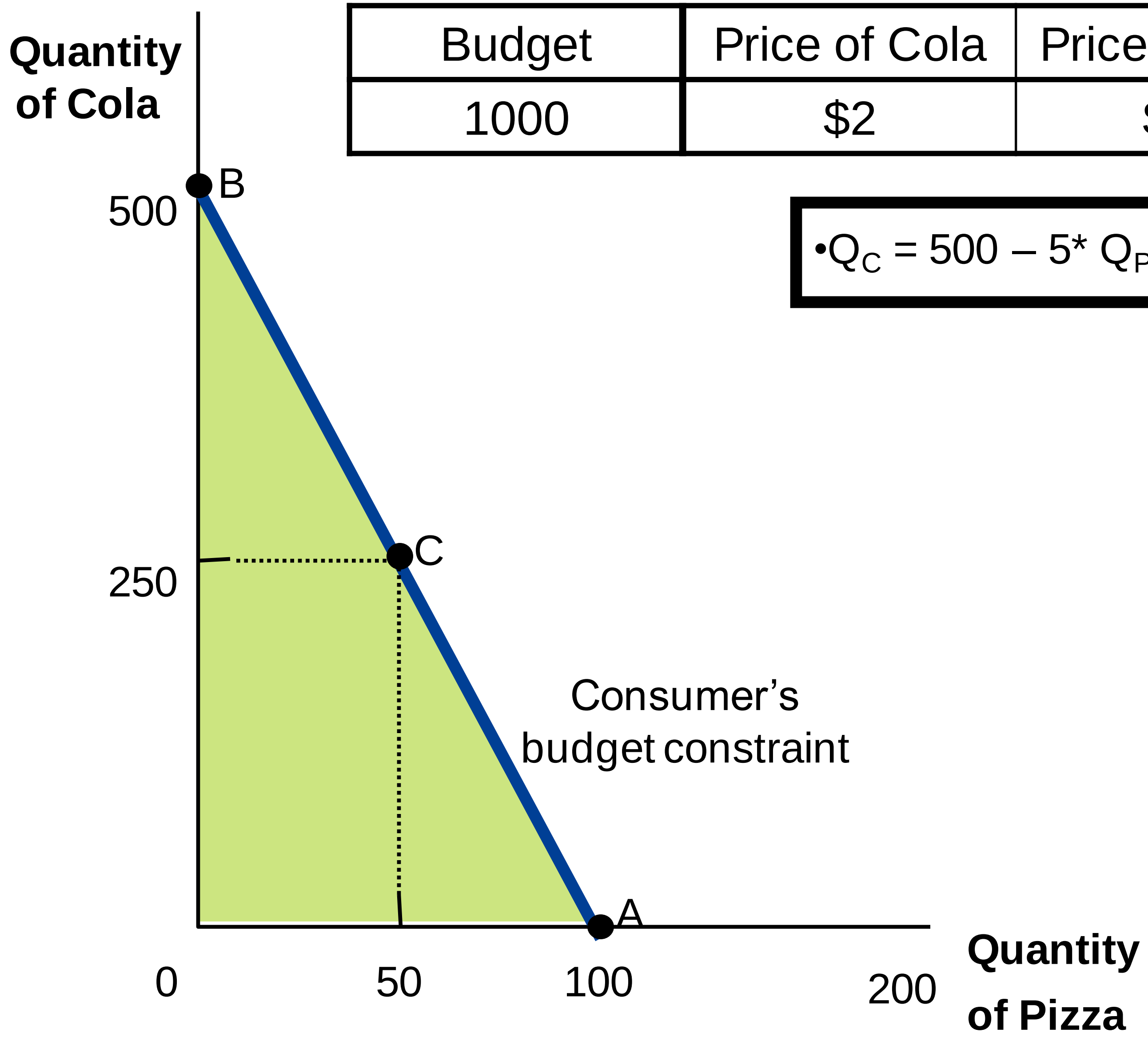
• $Q_C = 500 - 5 * Q_P$



The Consumer's Budget Constraint

Budget	Price of Cola	Price of Pizza
1000	\$2	\$10

• $Q_C = 500 - 5 * Q_P$



The Consumer’s Budget Constraint

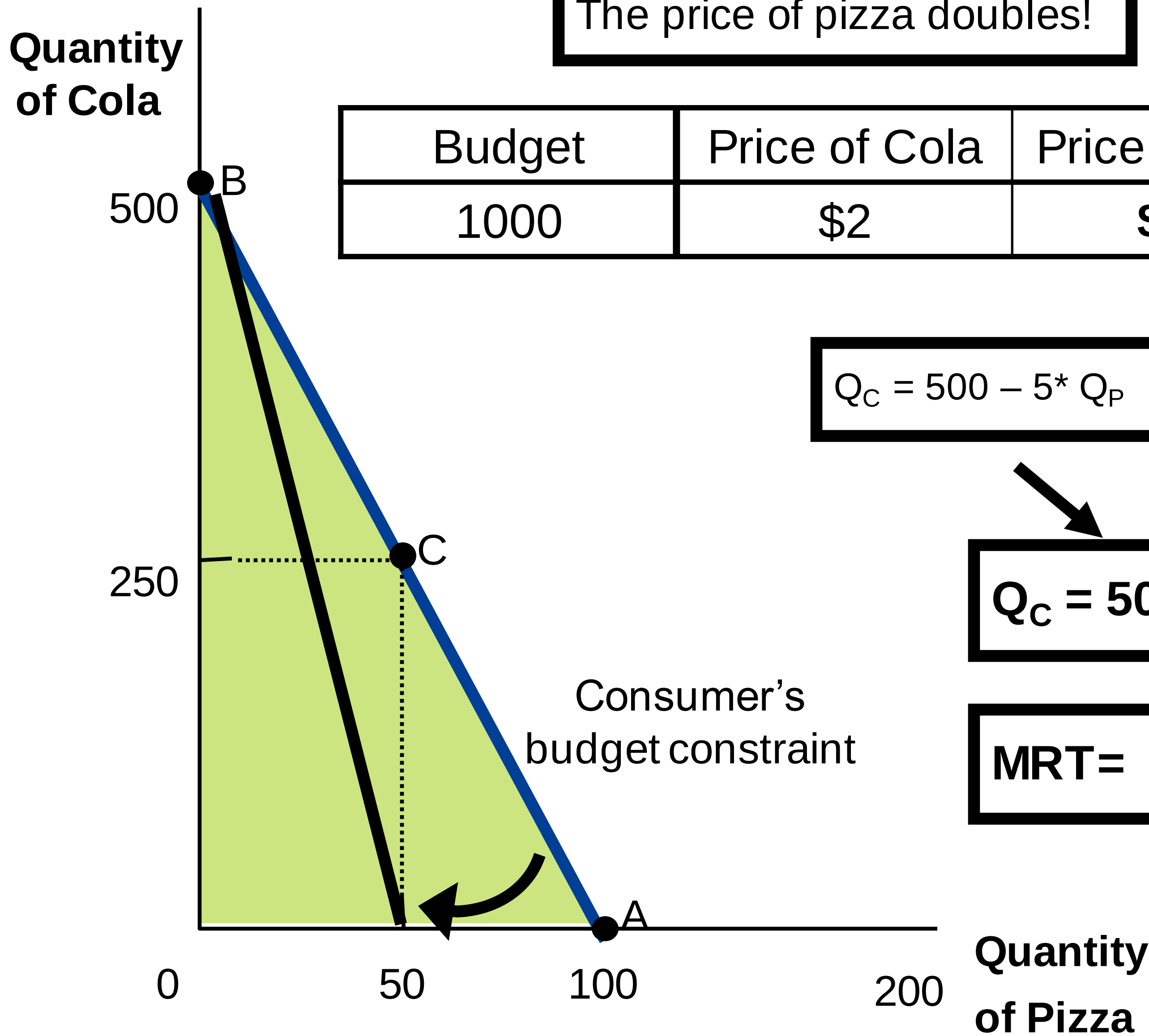
The price of pizza doubles!

Budget	Price of Cola	Price of Pizza
1000	$2	**$20**

$Q_C = 500 – 5^* Q_P$

$\mathbf{Q_C = 500 – 10^* Q_P}$

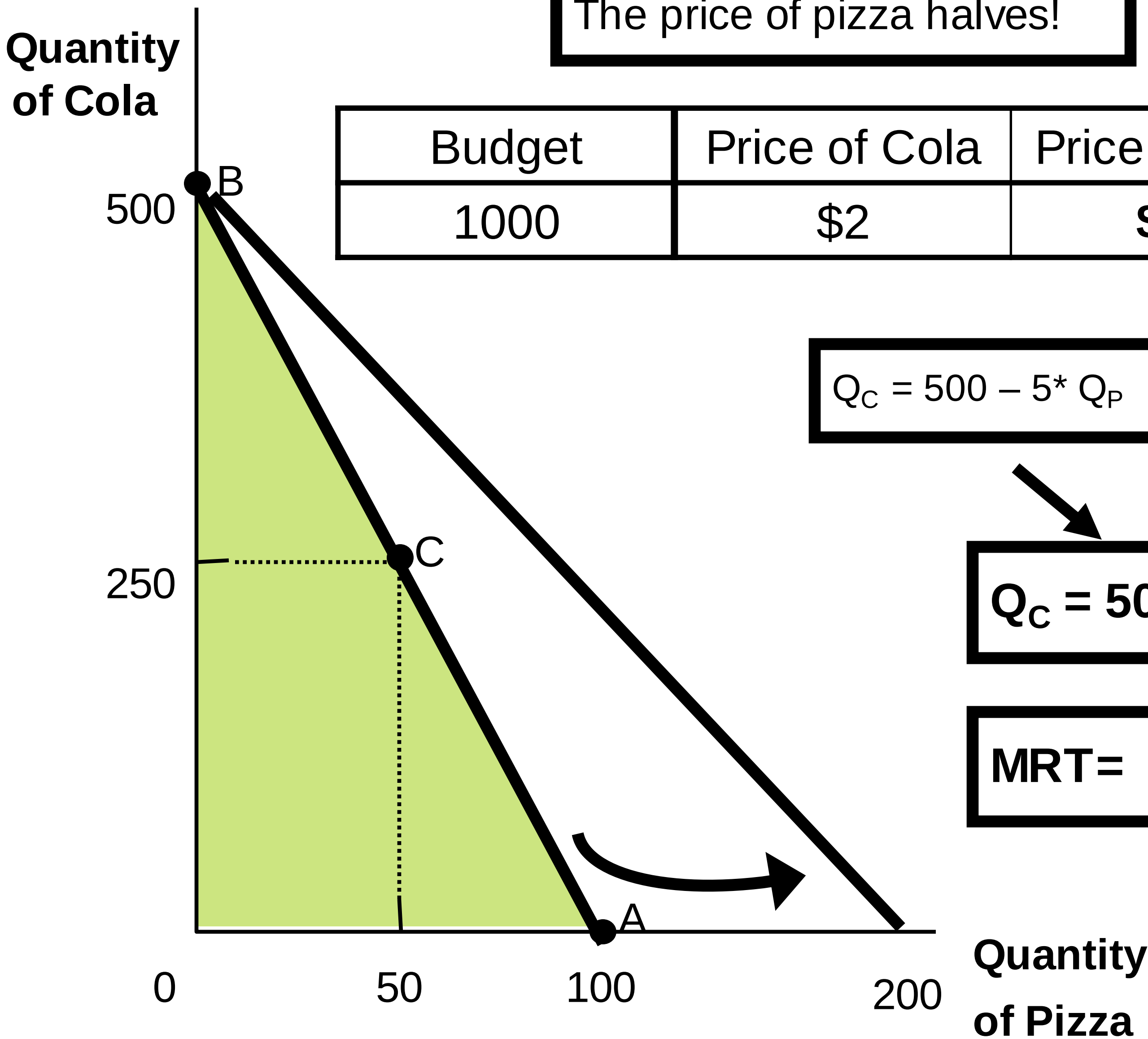
MRT= -10



The Consumer’s Budget Constraint

The price of pizza halves!

Budget	Price of Cola	Price of Pizza
1000	$2	**$5**



$Q_C = 500 – 5 * Q_P$

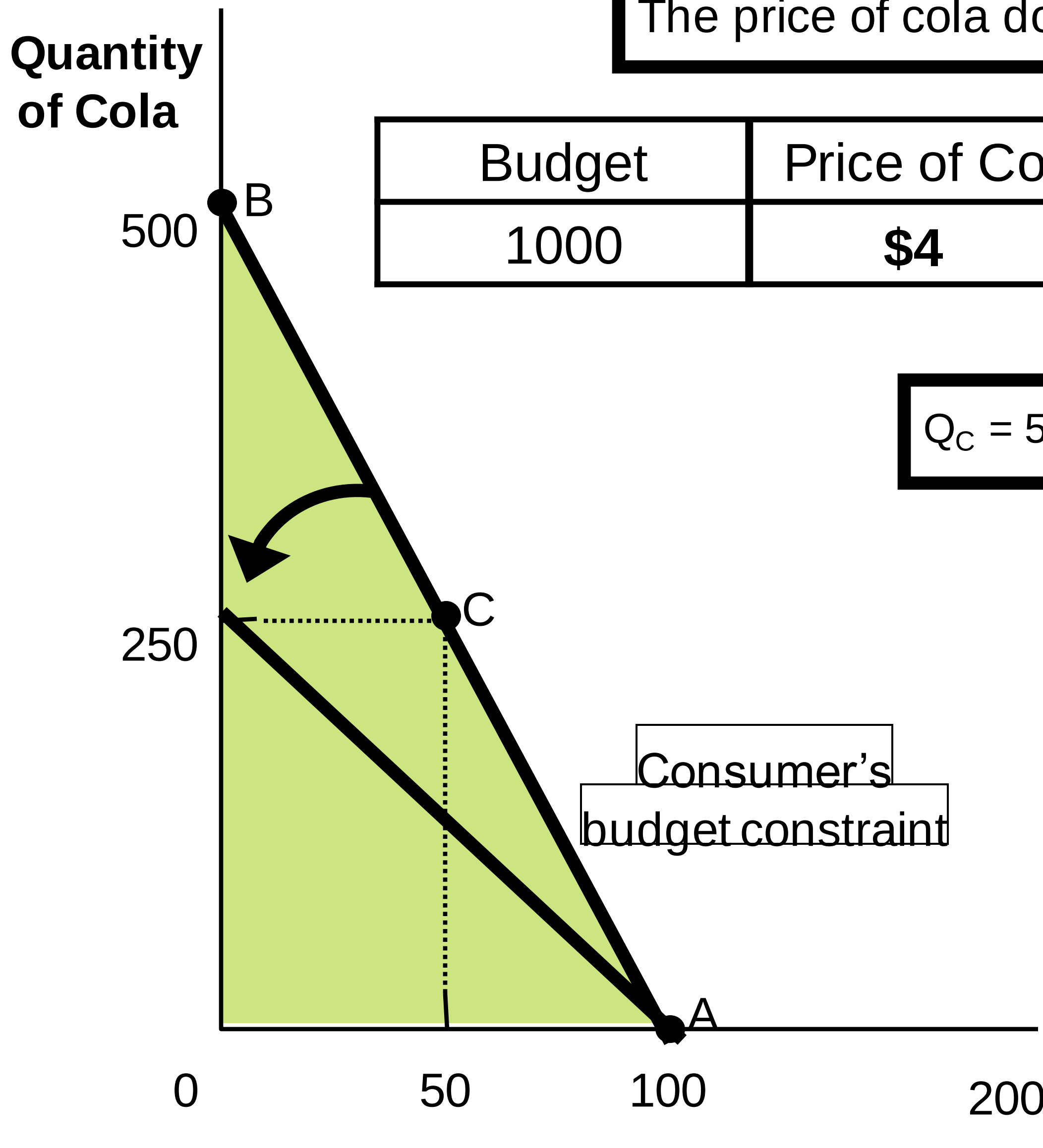
$$\mathbf{Q_C = 500 – 2½ * Q_P}$$

MRT= -2½

The Consumer’s Budget Constraint

The price of cola doubles!

Budget	Price of Cola	Price of Pizza
1000	**$4**	$10



$Q_C = 500 – 5^* Q_P$

$\mathbf{Q_C = 500 – 2½ * Q_P}$

MRT= -2½

The Consumer's Budget Constraint

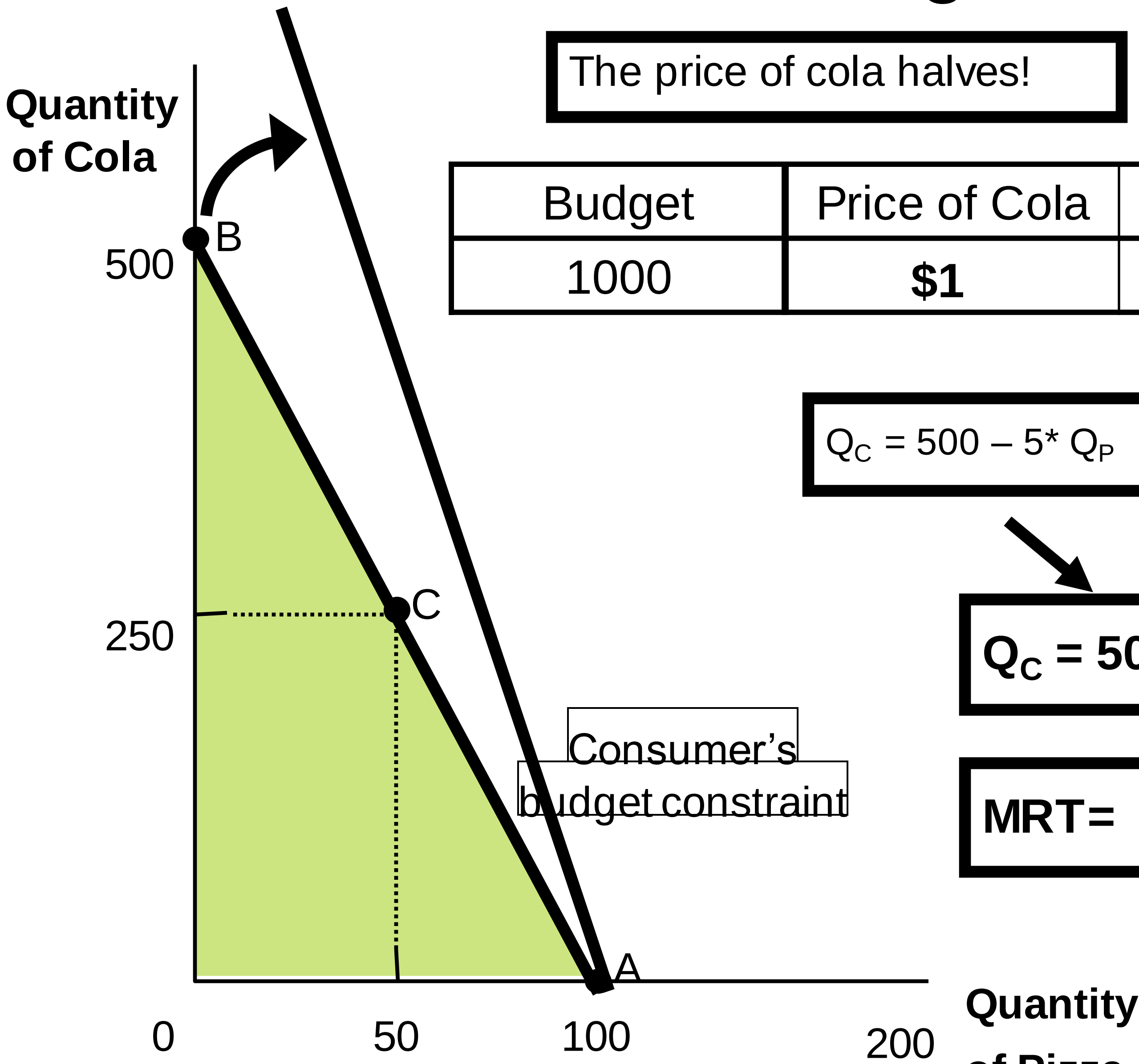
The price of cola halves!

Budget	Price of Cola	Price of Pizza
1000	**$1**	$10

$Q_C = 500 – 5 * Q_P$

$\mathbf{Q_C = 500 – 10 * Q_P}$

MRT= -10



The Consumer’s Budget Constraint

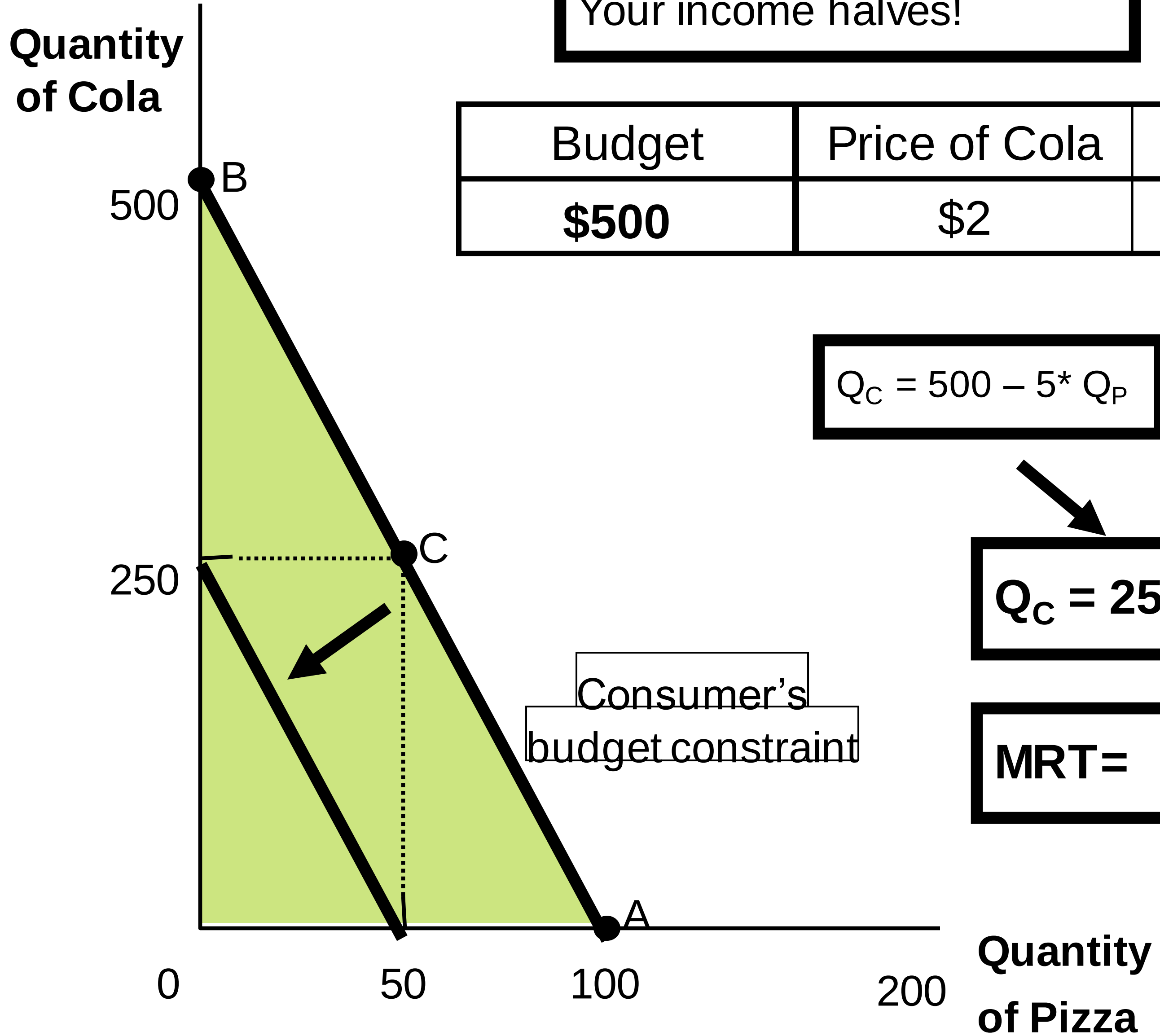
Your income halves!

Budget	Price of Cola	Price of Pizza
$500	$2	$10

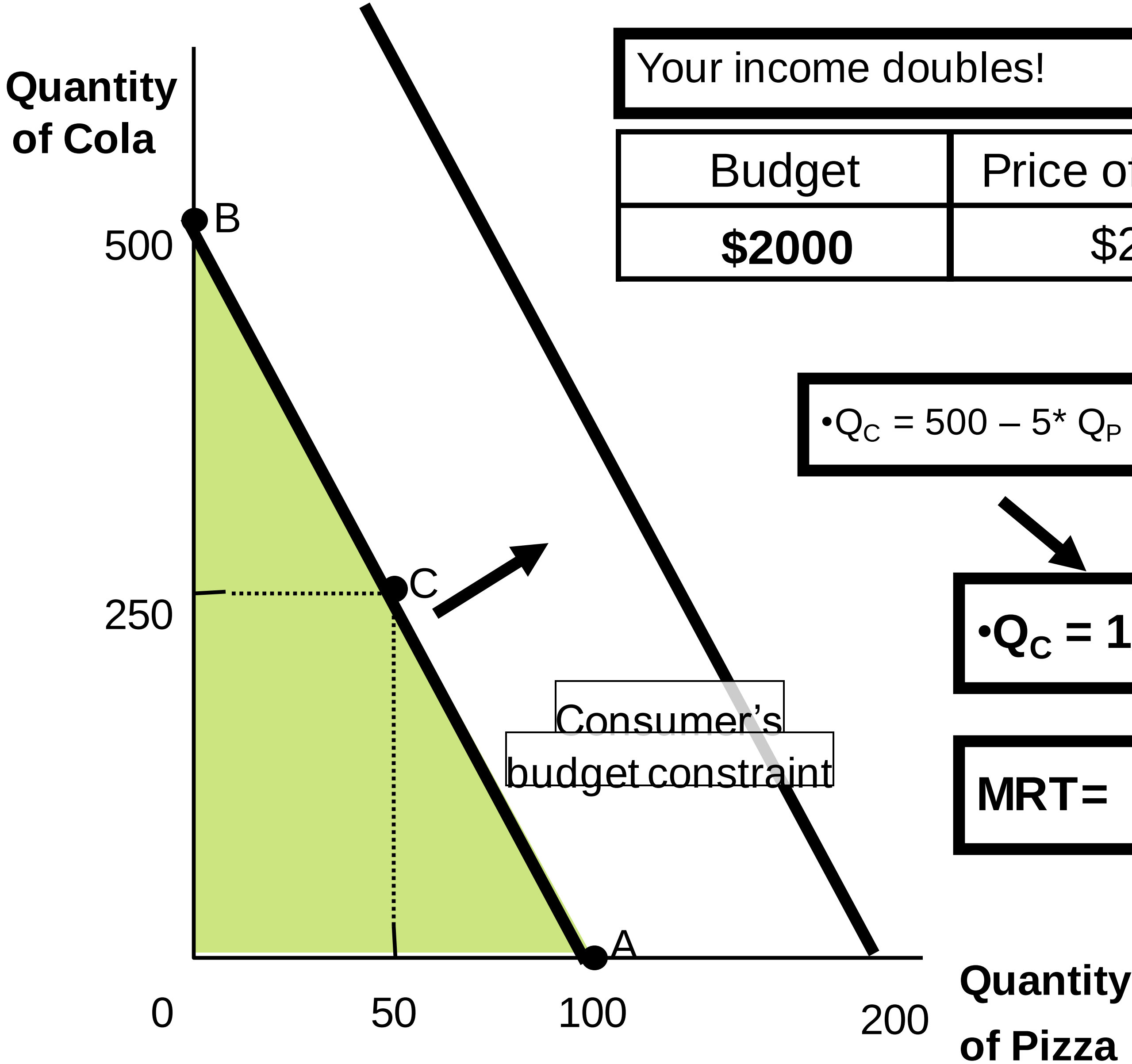
$Q_C = 500 – 5 * Q_P$

$\mathbf{Q_C = 250 – 5 * Q_P}$

MRT= -5



The Consumer's Budget Constraint



Your income doubles!

Budget	Price of Cola	Price of Pizza
$2000	$2	$10

- $Q_C = 500 - 5 * Q_P$

- **$Q_C = 1000 - 5 * Q_P$**

MRT= -10

How to find the slope of the budget restriction in the product space

1. **Write down the budget restriction** as $M = p1 * x1 + p2 * x2$
2. **Isolate x2**

   $p2 * x2 = M - p1 * x1$

   $x2 = (M - p1 * x1)/p2$

   $x2 = M/p2 - (p1/p2)\, x1$

3. **Differentiate x2 to x1**

   $dx2/dx1 = -(p1/p2)$

$$p_1x_1 + p_2x_2 = m$$

and

$$p_1(x_1 + \Delta x_1) + p_2(x_2 + \Delta x_2) = m.$$

Subtracting the first equation from the second gives

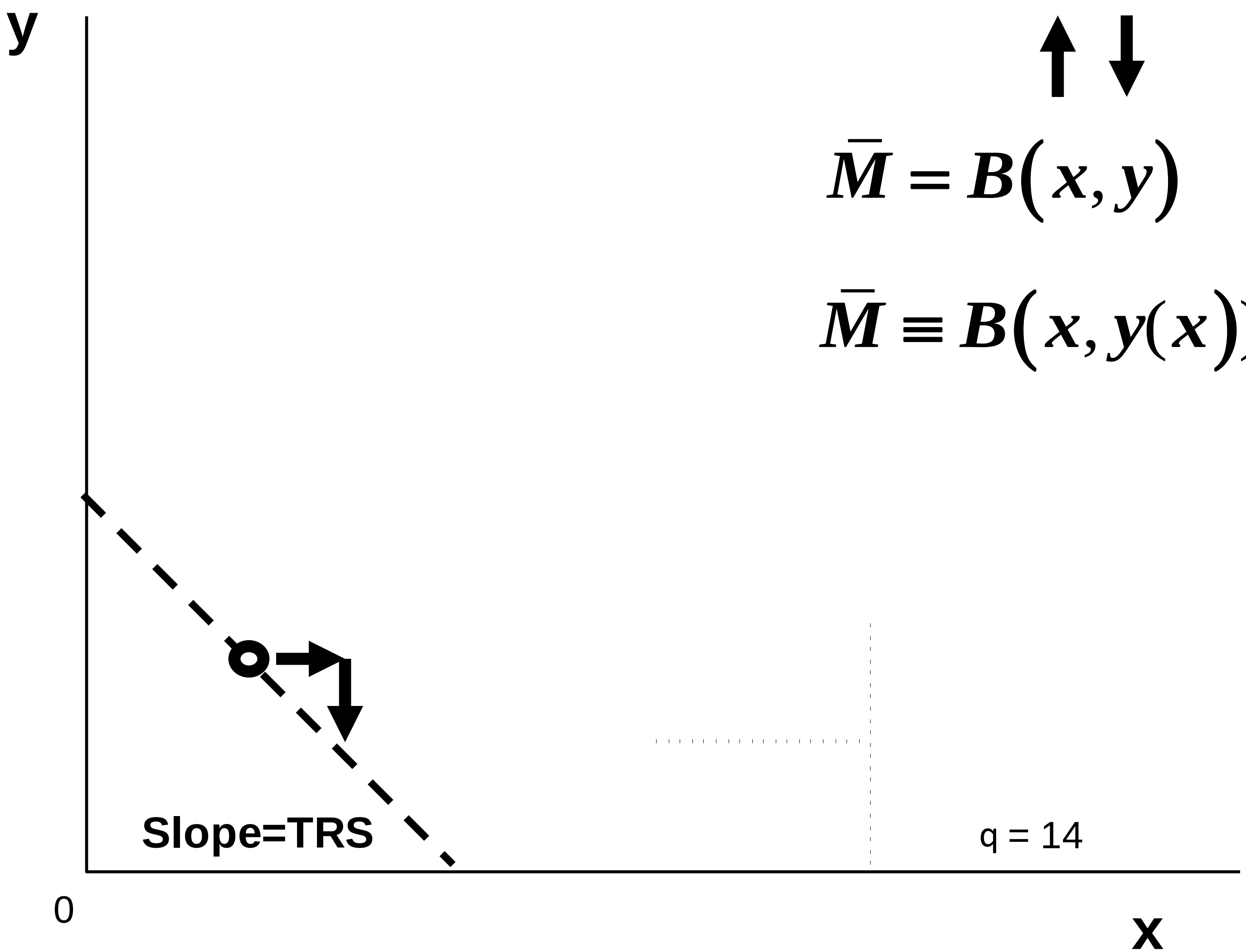
$$p_1\Delta x_1 + p_2\Delta x_2 = 0.$$

This says that the total value of the change in her consumption must be zero. Solving for $\Delta x_2/\Delta x_1$, the rate at which good 2 can be substituted for good 1 while still satisfying the budget constraint, gives

$$\frac{\Delta x_2}{\Delta x_1} = -\frac{p_1}{p_2}.$$

We look back on this method after talking about total differentiation

Budget restriction



$$\bar{M} = B(x, y)$$

$$\bar{M} \equiv B(x, y(x))$$

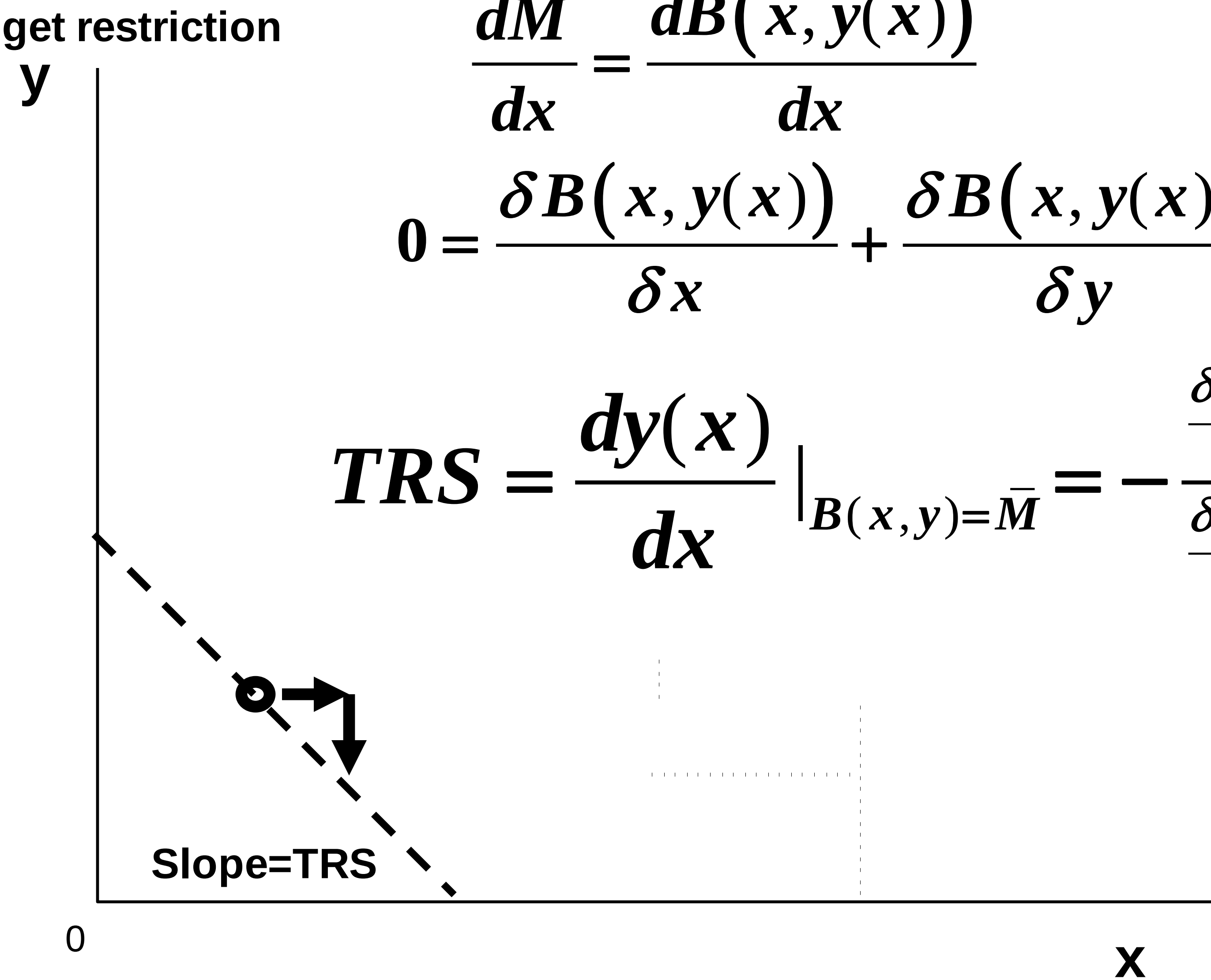
Derivation method 1: implicit function derivation

$$\bar{M} \equiv B(x, y(x))$$

$$\frac{d\bar{M}}{dx} = \frac{dB(x, y(x))}{dx}$$

$$0 = \frac{\delta B(x, y(x))}{\delta x} + \frac{\delta B(x, y(x))}{\delta y}\frac{dy}{dx}$$

$$TRS = \frac{dy(x)}{dx}\Big|_{B(x,y)=\bar{M}} = -\frac{\frac{\delta M(x,y)}{\delta x}}{\frac{\delta M(x,y)}{\delta y}}$$



2.5 The Numeraire

The budget line is defined by two prices and one income, but one of these variables is redundant. We could peg one of the prices, or the income, to some fixed value, and adjust the other variables so as to describe exactly the same budget set. Thus the budget line

$$p_1x_1 + p_2x_2 = m$$

is exactly the same budget line as

$$\frac{p_1}{p_2}x_1 + x_2 = \frac{m}{p_2}$$

or

$$\frac{p_1}{m}x_1 + \frac{p_2}{m}x_2 = 1,$$

- Exercises

REVIEW QUESTIONS V (Varian)

1. Originally the consumer faces the budget line $p_1x_1 + p_2x_2 = m$. Then
   - the price of good 1 doubles
   - the price of good 2 becomes 8 times larger
   - income becomes 4 times larger.

   Write down an equation for the new budget line in terms of the original prices and income.

2.1. The new budget line is given by $2p_1x_1 + 8p_2x_2 = 4m$.

2. What happens to the budget line if the price of good 2 increases, but the price of good 1 and income remain constant?

2.2. The vertical intercept (x_2 axis) decreases and the horizontal intercept (x_1 axis) stays the same. Thus the budget line becomes flatter.

REVIEW QUESTIONS V (Varian)

3. If the price of good 1 doubles and the price of good 2 triples, does the budget line become flatter or steeper?

2.3. Flatter. The slope is $-2p_1/3p_2$.

5. Suppose that the government puts a tax of 15 cents a gallon on gasoline and then later decides to put a subsidy on gasoline at a rate of 7 cents a gallon. What net tax is this combination equivalent to?

2.5. A tax of 8 cents a gallon.

REVIEW QUESTIONS V (Varian)

6. Suppose that a budget equation is given by $p_1x_1+ p_2x_2 =m$.

   The government decides to impose:

   – a lump-sum tax of u
   – a quantity tax on good 1 of t
   – a quantity subsidy on good 2 of s.

   What is the formula for the new budget line?

2.6. $(p_1 + t)x_1 + (p_2 - s)x_2 = m - u.$

7. If the income of the consumer increases and one of the prices decreases at the same time, will the consumer necessarily be at least as well-off?

2.7. Yes, since all of the bundles the consumer could afford before are affordable at the new prices and income.

Consider a version of the consumer problem in which quasilinear utility $x_1^{\frac{1}{2}} + \frac{1}{4}x_2$ is maximised subject to $x_1 + x_2 = 1$.

$$L = x_1^{.5} + \frac{1}{4}x_2 + \lambda \cdot \left(1 - (x_1 + x_2)\right)$$

$$0 = L_1 = .5 \cdot x_1^{-.5} - \lambda \quad \Leftrightarrow x_1 = \left(\frac{1}{2\lambda}\right)^2$$

$$0 = L_2 = \frac{1}{4} - \lambda \quad \Leftrightarrow \lambda = \frac{1}{4}$$

Thus then $x_1 = 4$

$$0 = L_\lambda = 1 - (x_1 + x_2)$$

Thus then $x_2 = 1 - x_1 = -3$

In a consumer problem, negative quantities are not possible

Consider a version of the consumer problem in which quasilinear utility $x_1^{\frac{1}{2}} + \frac{1}{4}x_2$ is maximised subject to $x_1 + x_2 = 1$.

$$L = x_1^{.5} + \frac{1}{4}x_2 + \lambda \cdot \left(1 - (x_1 + x_2)\right)$$

$$0 = x_1 \cdot L_1 = x_1 \cdot \left(.5 \cdot x_1^{-.5} - \lambda\right) \Leftrightarrow x_1 = \left(\frac{1}{2\lambda}\right)^2 = \left(2\lambda\right)^{-2} \text{ or } x_1 = 0$$

$$0 = x_2 \cdot L_2 = x_2 \cdot \left(\frac{1}{4} - \lambda\right) \qquad \Leftrightarrow \lambda = \frac{1}{4} \text{ or } x_2 = 0$$

$$0 = L_\lambda = 1 - (x_1 + x_2)$$

Suppose $x_1 = 0, x_2 > 0$: Then $x_2 = 1, L = \frac{1}{4}$

Worrysome is that $.5x_1^{-.5}$ is infinitely large and >0 when $x_1 = 0$

Suppose $x_1 > 0, x_2 = 0$: Then $x_1 = 1, L = 1$

- REVIEW QUESTIONS N (Nicholson)

2.7

Consider the following constrained maximization problem:

$$\begin{array}{ll} \textbf{maximize} & y = x_1 + 5 \ln x_2 \\ \textbf{subject to} & k - x_1 - x_2 = 0, \end{array}$$

where k is a constant that can be assigned any specific value.

a. Show that if $k = 10$, this problem can be solved as one involving only equality constraints.

b. Show that solving this problem for $k = 4$ requires that $x_1 = -1$.

c. If the x's in this problem must be nonnegative, what is the optimal solution when $k = 4$?

d. What is the solution for this problem when $k = 20$? What do you conclude by comparing this solution to the solution for part (a)?

Note: This problem involves what is called a "quasi-linear function." Such functions provide important examples of some types of behavior in consumer theory—as we shall see.

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