Game Theory

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MLÁDEŽE A TĚLOVÝCHOVY

Task for you

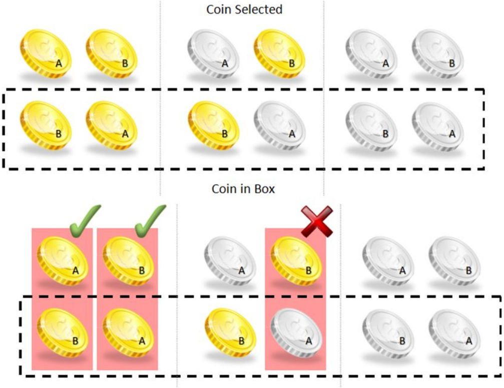
Suppose there is a certain disease randomly found in one-half of one percent (.005) of the general population. A certain clinical blood test is 99 percent (.99) effective in detecting the presence of this disease; that is, it will yield an accurate positive result in 99 percent of the cases where the disease is actually present. But it also yields false-positive results in 5 percent (.05) of the cases where the disease is not present.

What is the probability that some person has this disease when the test is positive?

Another task

- The boxes are shuffled so that you don't know which is which. You select a random box and, without looking inside, reach in and pull out one of the coins at random. You close the lid after selection, then look at the coin you picked. It is a gold coin.
- Question: *What is the probability that the other coin in the box is also gold?*

Coins solution: intuitive



Out of all six (equally likely) ways that the coins can be arranged we can see that, in two of the three cases when a gold coin is selected, the other coin is also gold. Therefore the probability that the other coin is gold is 2/3.

Bayes' rule

- Bayes rule gives the conditional probability of an event when another event has been observed, i.e., it gives us a criterion to determine how new information should change our beliefs about a given event

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- Let us write $p(A|B)$ the probability of the event A when B has been observed

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- Let p(A) and p(B) two a priori probability of the events A and B
- Let us write p(A|B) the probability of the event A when B has been observed

- **Bayes rule is a formula for determining p(A|B)**

Bayes' rule

$$p(A|B) = \frac{p(A) * p(B|A)}{p(B)}$$

- where: p(A) is the a priori probability of A before occurring B, p(B|A) is the conditional probability of B given A, and p(B) is the a priori probability of B
- Note that by definition, p(B) includes all the possible situations in which B can verify itself, irrespective the presence/absence of A.
- Therefore:

$$p(B) = p(A)p(B|A) + p(\neg A)p(B|\neg A)$$

Combining both expressions...

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Example

- You are on the train and you want to understand if the person sitting next to you is a center-right
- You know a priori that 54% of the citizens are center-right voters (46% center-left)
- Now the person sitting next to you open a newspaper. You know that the 35% of center-right voters read that newspaper (while it is read by 65% of center-left voters)
- What is your update belief that the person sitting next to you is a center-right voter?

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- What is your update belief that the person sitting next to you is a center-right voter?

$$p(CR|N) = p(CR)\ p(N|CR)/(p(CR)\ p(N|CR)+p(CL)\ p(N|CL)) = .54{*}.35/(.54{*}.35+.46{*}.65) = .387$$

General rule

More in general, given $(h_1, h_2, \ldots, h_n)$ a set of mutually exclusive and exhaustive events compatible with the event k, then:

$$p(h_1|k) = \frac{p(h_1) * p(k|h_1)}{\sum_{i=1}^{n} p(h_i)p(k|h_i)}$$

Belief updating

$$p(h_1|k) = \frac{p(h_1) * p(k|h_1)}{\sum_{i=1}^{n} p(h_i)p(k|h_i)}$$

- This is an extremely important formula, called **Bayes' rule**, which enables us to calculate the posterior probability about an event based on the prior probability and new information/evidence.

Belief updating

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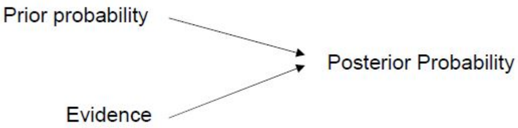
- This is an extremely important formula, called **Bayes' rule**, which enables us to calculate the posterior probability about an event based on the prior probability and new information/evidence.

- Prior belief: a player's initial belief about the probability of an event (i.e., $p(h_1)$).
- Posterior belief: a player's updated belief after receiving new information/evidence (i.e., $p(h_1|k)$).

Bayes' rule



- Bayes rule tells us how the available evidence should alter our belief in something being true

Another task

- The boxes are shuffled so that you don't know which is which. You select a random box and, without looking inside, reach in and pull out one of the coins at random. You close the lid after selection, then look at the coin you picked. It is a gold coin.
- Question: *What is the probability that the other coin in the box is also gold?* **Use Bayes' rule**

Let's continue...

- What is the probability that the other coin in the box is silver?

Let's continue...

- What is the probability that the other coin in the box is silver?
- What if we return coin back to the box, shuffle the coins and in a same way (without looking into the box) we draw another coin.
    - *It is gold again. What is the probability that the other coin in the box is also gold?* **Use Bayes' rule**

Task for you

Suppose there is a certain disease randomly found in one-half of one percent (.005) of the general population. A certain clinical blood test is 99 percent (.99) effective in detecting the presence of this disease; that is, it will yield an accurate positive result in 99 percent of the cases where the disease is actually present. But it also yields false-positive results in 5 percent (.05) of the cases where the disease is not present.

What is the probability that some person has this disease when the test is positive? ***Now, you can calculate it.***

Bayesian updating

$$P(disease|positive) = \frac{P(positive|disease) * P(disease)}{P(positive)} =$$

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$$P(disease|positive) = \frac{P(positive|disease) * P(disease)}{P(positive)} =$$

$$= \frac{P(positive|disease) * P(disease)}{P(positive|disease) * P(disease) + P(positive|healthy) * P(healthy)} = 0.0905$$

Bayesian games

- "Bayesian games": information is incomplete and asymmetric
  – at least one of the players is unsure about the other's pay-offs.
- Bayesian Nash equilibrium: the Nash equilibrium concept taking in account players beliefs
  – Beliefs and strategies must be consistent
    1. proposing a strategy combination
    2. calculating the beliefs generated by those strategies.
    3. Check that the proposed strategies are optimal given the players' beliefs.
    4. *Check that the players' strategies are optimal, that is best responses to each other and consistent with their beliefs:*

Party-loving or hotel-loving stalker?

Matrix 7.1.1 Mr Column is a party lover

Mr Column PL

		party	hotel
Ms Row	party	2, 3	4, 0
	hotel	3, 2	0, 1

Matrix 7.1.2 Mr Column is not a party-lover

Mr Column HL

		party	hotel
Ms Row	party	2, 1	4, 2
	hotel	3, 0	0, 3

Party-loving or hotel-loving stalker?

- Uncertainty about the type of Mr. Column:
  - P = Pr(Mr. Column=PL)
  - 1-P = Pr(Mr. Column=HL)
- What, given P, would be the strategy of Ms Row?

Party-loving or hotel-loving stalker?

Step 1: a strategy for Mr Column conditional on his type:

- The party-loving Mr Column: choose party.
- The non-party-loving Mr Column: choose hotel.

Step 2: Calculate beliefs for Ms Row that are consistent with Mr Column's strategy as defined in Step 1 and probability, P, that Mr Column is a party lover.

- Consistent beliefs for Ms Row are that:
  - Probability P: Mr Column is a party lover and chooses party.
  - Probability (1 – P): Mr Column is not a party lover and chooses hotel.

Step 3: Strategy for Ms Row that is consistent with her beliefs about Mr Column.

- Find the outcome with highest payoff:
  - Choose hotel if EPOhotel > EPOparty.
  - Choose party if EPOparty > EPOhotel
- Payoff is dependend on the value of P. Find critical value of P* where
  - EPOhotel > EPOparty if P > P*
  - EPOhotel < EPOparty if P < P*
- Strategy for Ms Row now becomes:
  - Choose hotel if P > P*
  - Choose party if P < P*

Step 4*: Check that the players' strategies are optimal, that is best responses to each other and consistent with their beliefs:*

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	hotel	3, 0	0, 3

$$EPO_{hotel} = P \cdot 3 + (1 - P) \cdot 0 = 3P$$

$$EPO_{party} = P \cdot 2 + (1 - P) \cdot 4 = 4 - 2P$$

$$\begin{aligned} EPO_{hotel} > EPO_{party} &\leftrightarrow 3P^* > 4 - 2P^* \\ &\leftrightarrow 5P^* > 4 \\ &\leftrightarrow P^* > \tfrac{4}{5} = 0.8 \end{aligned}$$

Party-loving or hotel-loving stalker?

- Strategy for Ms Row:
  - Choose hotel if P > P*
  - Choose party if P < P*
- now becomes
  - Choose hotel if P > 0.8
  - Choose party if P < 0.8

Step 4: Check that the players' strategies are optimal, that is best responses to each other and consistent with their beliefs:

- Mr Column
  - Party-lover: chooses party.
  - Non-party-lover: chooses hotel.

- Ms Row
  - If P > 0.8 choose hotel.
  - If P < 0.8 choose party.
  - If P = 0.8 randomise.

Party-loving or hotel-loving stalker?

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Mr Column HL

Ms Row		party	hotel
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- Having avoided the stalker, has Mrs.Row a slight preference for party or for hotel?
  - Hotel (4 versus 3)
- Exercise: what if Mrs.Row has a slight preference for party (without the stalker)?

Party-loving or hotel-loving stalker?

Ms Row has a slight preference for party

Matrix 7.1.1 Mr Column is a party lover

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Mr Column HL

Ms Row		party	hotel
	party	2, 1	4, 2
	hotel	3, 0	0, 3

Ms Row has a slight preference for hotel

Matrix 7.2.1 Mr Column is a party-lover

Mr Column

Ms Row		party	hotel
	party	0, 3	3, 0
	hotel	4, 2	2, 1

Matrix 7.2.2 Mr Column is not a party-lover

Mr Column

Ms Row		party	hotel
	party	0, 1	3, 2
	hotel	4, 0	2, 3

$$EPO_{hotel} = P \cdot 4 + (1-P) \cdot 2 = 2 + 2P$$

$$EPO_{party} = P \cdot 0 + (1-P) \cdot 3 = 3 - 3P$$

$$EPO_{hotel} > EPO_{party} \leftrightarrow 2 + 2P^* > 3 - 3P^*$$

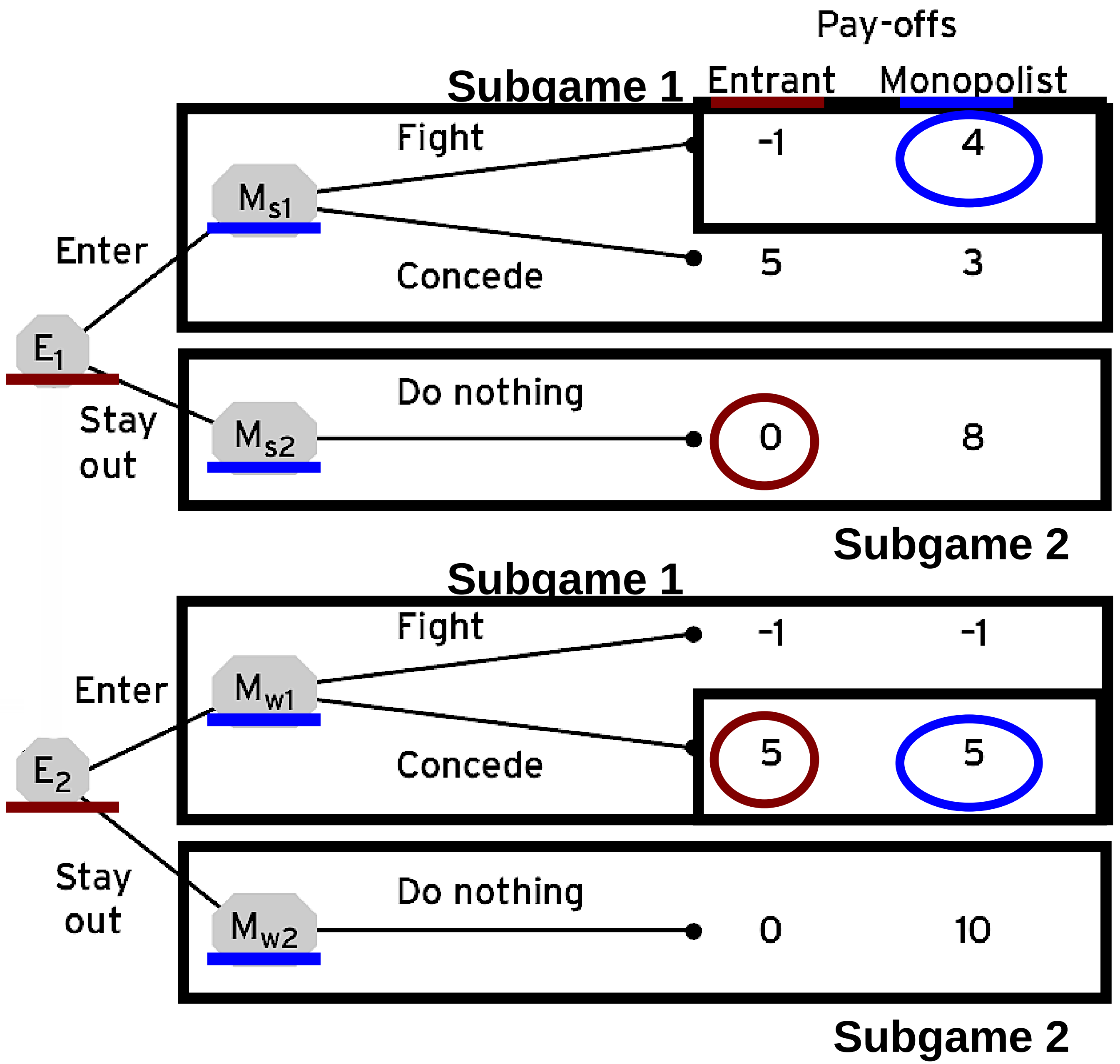
$$\leftrightarrow 5P^* > 1$$

$$\leftrightarrow P^* > \tfrac{1}{5} = 0.2$$

Entry deterrence with incomplete information

Two possibilities:

- A strong or a weak monopolist
- Which of two is the strong monopolist: the upper or the lower?
  - The upper is the strong monopolist as it will fight when the entrant enters



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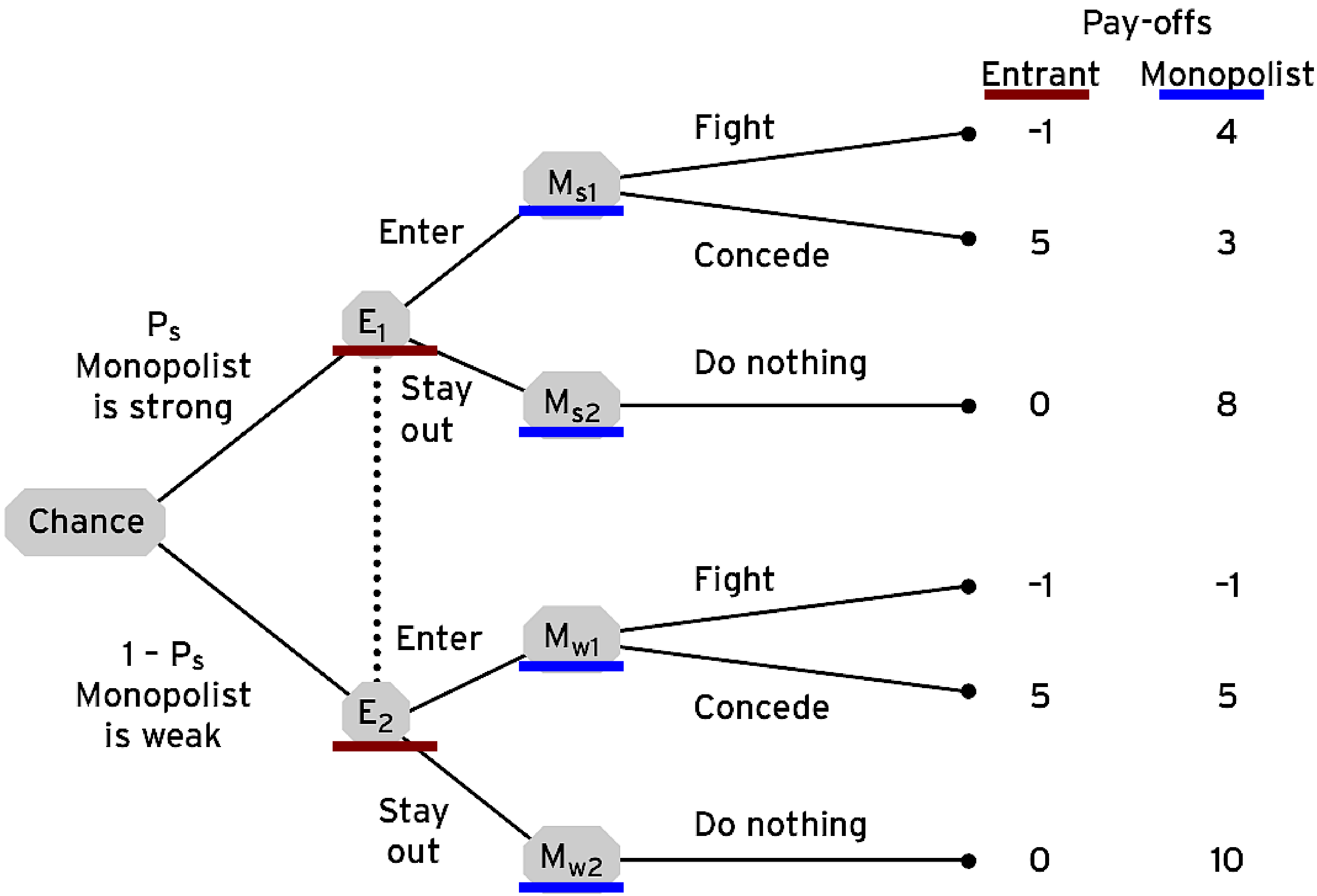

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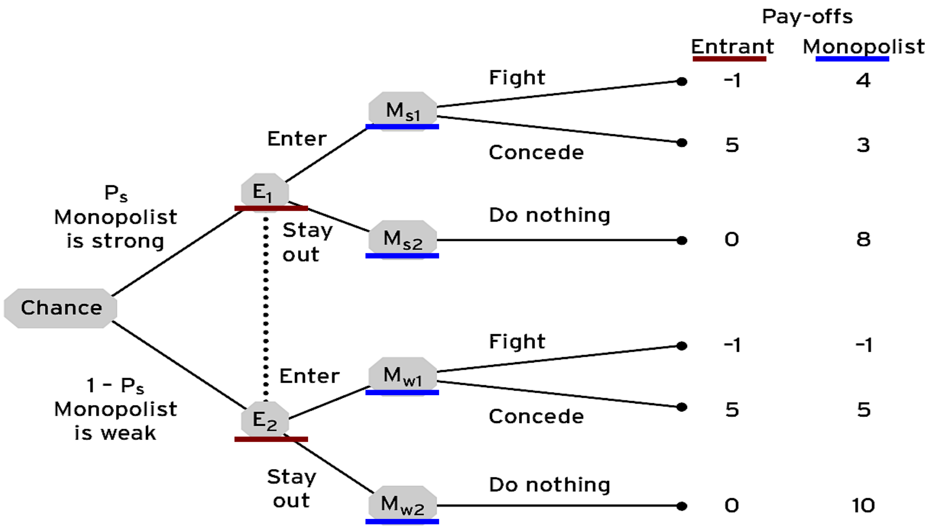

$$EPO_{Entry} = P_{Strong} \cdot -1 + (1 - P_{Strong}) \cdot 5 = 5 - 6P_{Strong}$$

$$EPO_{Stay\ out} = P_{Strong} \cdot 0 + (1 - P_{Strong}) \cdot 0 = 0$$

$$EPO_{Entry} > EPO_{Stay\,out} \leftrightarrow 5 - 6P^{*}_{Strong} > 0$$

$$\leftrightarrow P^{*}_{Strong} < \tfrac{5}{6}$$

So if P*strong < 5/6, enter, otherwise stay out of the market

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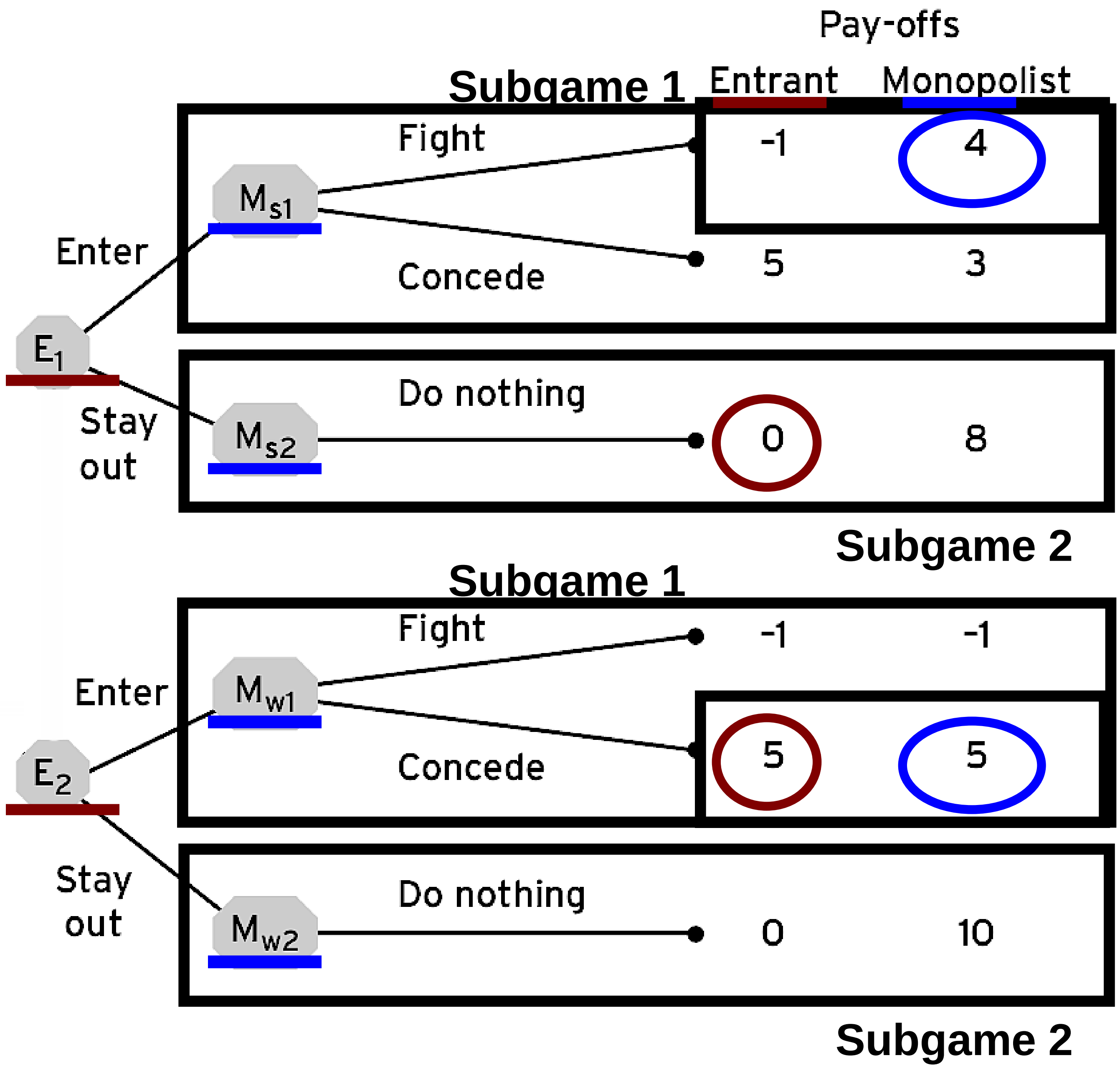

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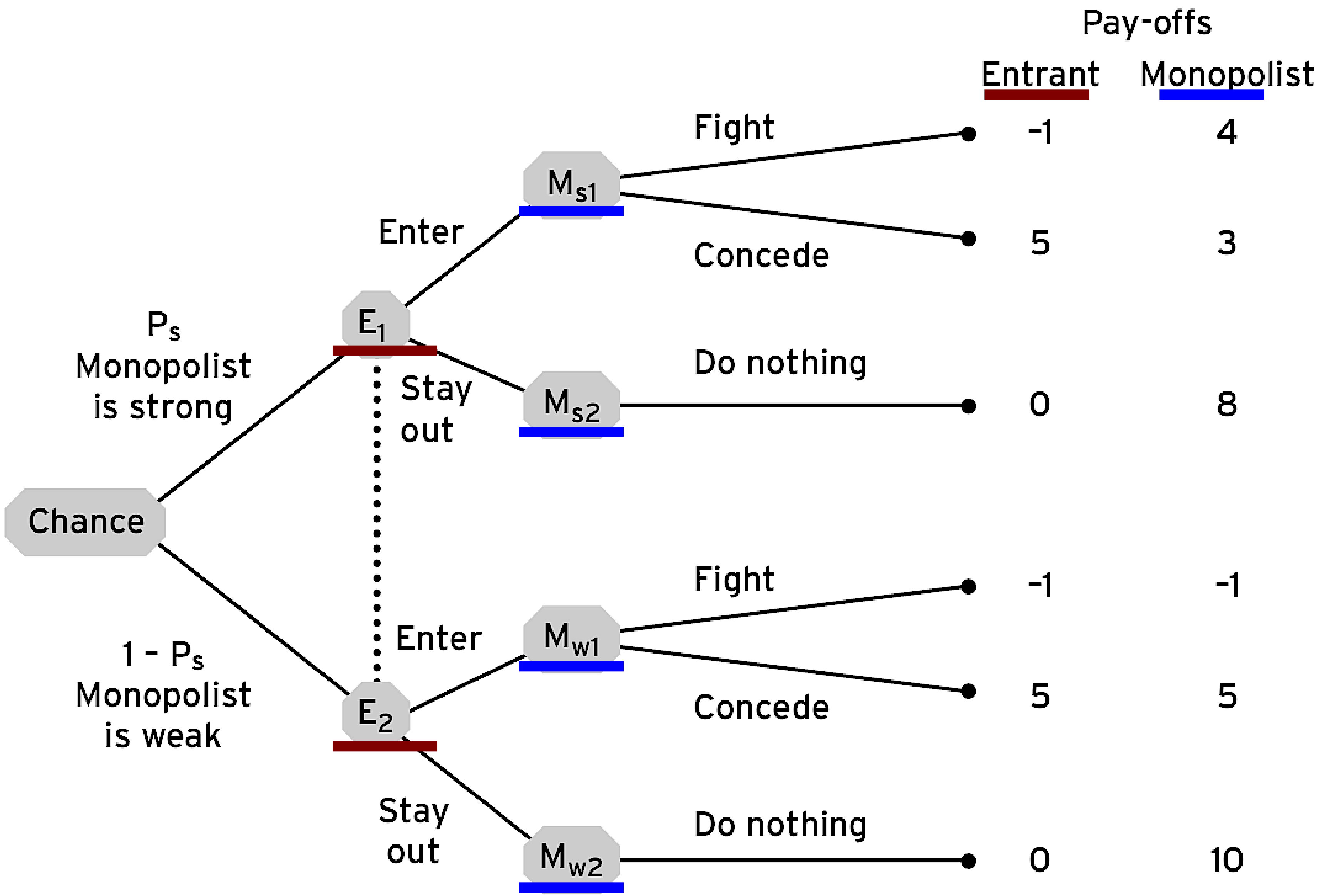

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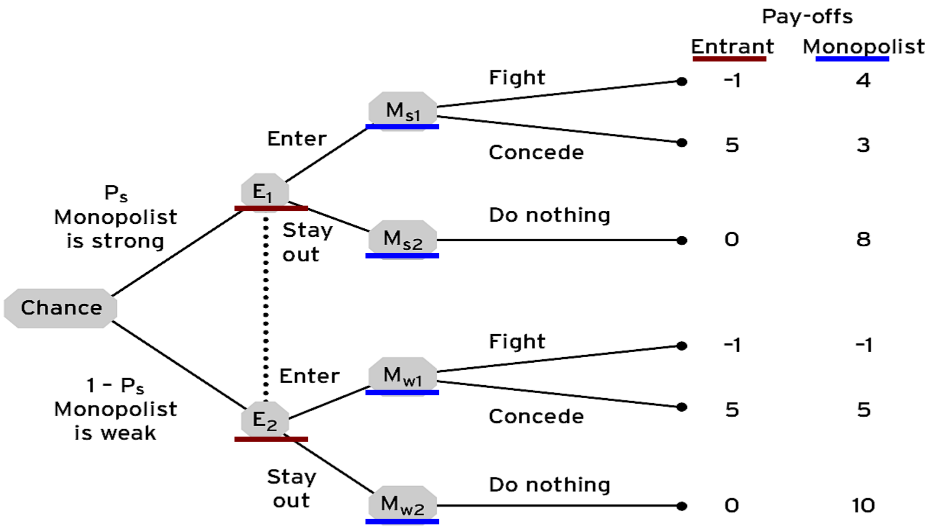


$$EPO_{Entry} = P_{Strong} \cdot -1 + (1 - P_{Strong}) \cdot 5 = 5 - 6P_{Strong}$$

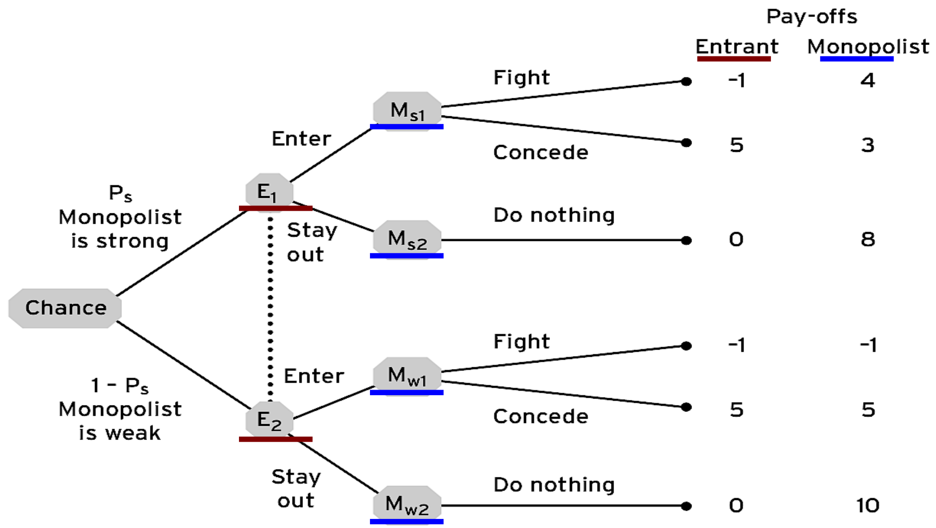
$$EPO_{Stay\,out} = P_{Strong} \cdot 0 + (1 - P_{Strong}) \cdot 0 = 0$$

$$EPO_{Entry} > EPO_{Stay\,out} \leftrightarrow 5 - 6P^*_{Strong} > 0$$

$$\leftrightarrow P^*_{Strong} < \tfrac{5}{6}$$

So if P*strong < 5/6, enter, otherwise stay out of the market

Entry deterrence with incomplete information



So if P*strong < 5/6, enter, otherwise stay out of the market

Imagine the following case: $P^*_{Strong} = \frac{1}{2} < \frac{5}{6} \approx 0.83$

- What happens when you are a strong monopolist? Do entrants enter?
  - Yes! Entrants do not know that you are strong! Just that Pstrong= ½ < 5/6. The best action for an entrant is to enter
- Is there anything you can do?
  - How about trying to signal them that you are a strong monopolist?


Signal success
Well-groomed
Always looking cool
Expensive suit
American Psycho
Slim, muscled body

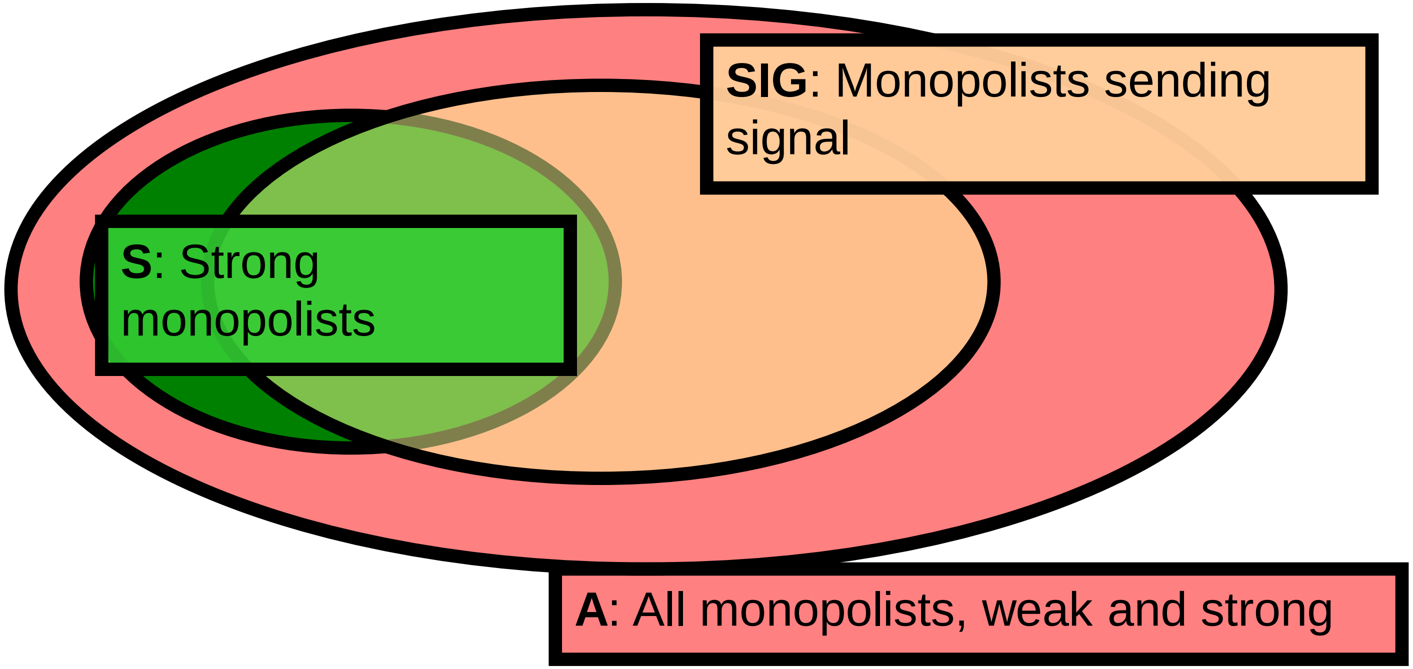
Signal success

Signal wealth

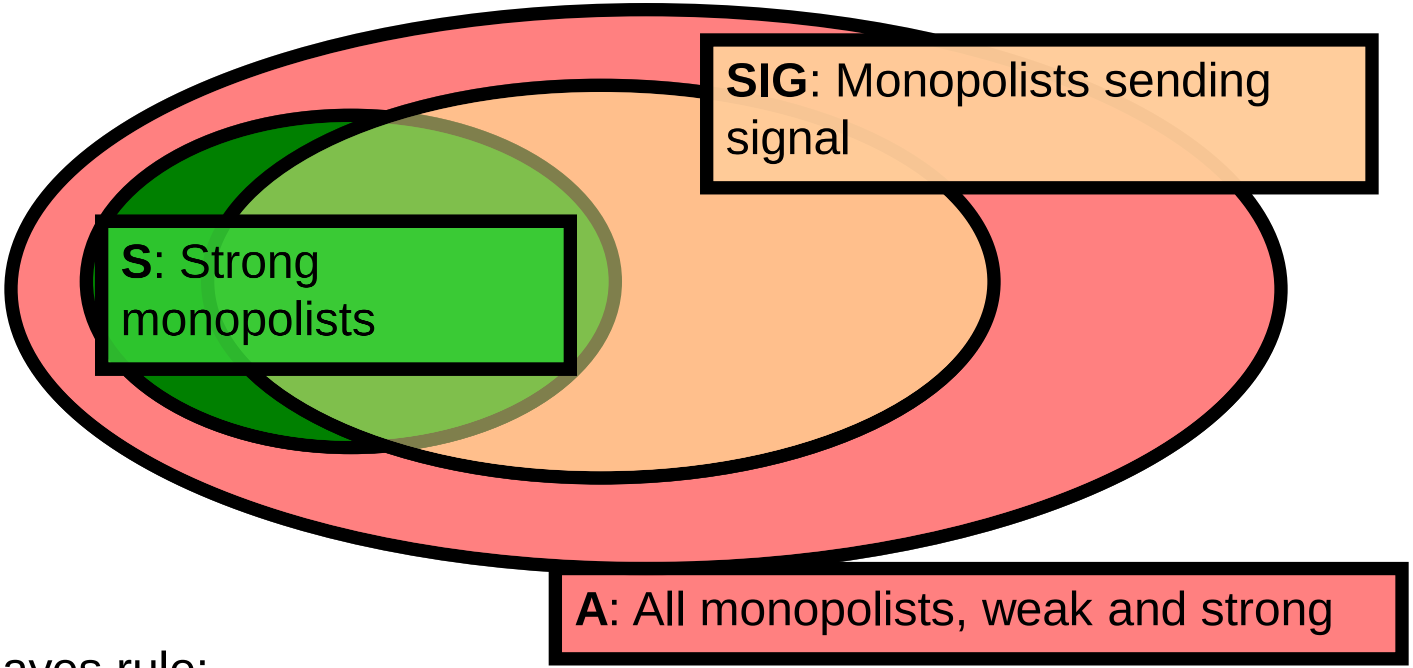
Entry deterrence with incomplete information

Signaling

- How can the strong monopolist signal his strength?
  - Message: “I am a strong monopolist”
  - But this can easily be copied by a weak monopolist: cheap talk
- The signal must be easy for a strong and difficult for a weak monopolist
  - Targeted marketing or publicity campaign
  - Make a special offer to its existing customers.
  - Burn money!



- What is the proportion of strong monopolists?
  - S/A
- What is the proportion of monopolists sending a signal?
  - SIG/A
- What is the proportion of strong monopolists sending a signal?
  - (S&SIG)/SIG
- What is the probability that a monopolist is strong after I have seen a signal?
  - (S&SIG)/SIG
- Does signaling help?
  - When (S&SIG)/SIG > S/A
  - When the signal is ***informative.***



Bayes rule:

$$P[S \mid SIG] = \frac{P[S \,\&\, SIG]}{P[\mathrm{SIG}]}$$

$$\Leftrightarrow P[S \mid SIG] = P[SIG \mid S] \cdot \frac{P[S]}{P[\mathrm{SIG}]}$$

$$P[SIG \mid S] = \frac{P[SIG \,\&\, S]}{P[S]}$$

$$= \frac{P[S \,\&\, SIG]}{P[\mathrm{S}]}$$

$$P[S \,\&\, SIG] = P[SIG \mid S] \cdot P[S]$$

Where $P[SIG] = P[S] \cdot P[SIG \mid S] + P[\mathrm{W}] \cdot P[SIG \mid \mathrm{W}]$

Entry deterrence with incomplete information

- Original believe that monopolist is strong: P_{Strong}
- After signal (SIG):

$$P_{Strong}^{UPD} = \Pr[M_{Strong} \mid SIG] = \Pr[SIG \mid M_{Strong}] \cdot \frac{P_{Strong}}{\Pr[SIG]}$$

$$= \Pr[SIG \mid M_{Strong}] \cdot \frac{P_{Strong}}{\Pr[SIG \mid M_{Strong}] \cdot P_{Strong} + \Pr[SIG \mid M_{Weak}] \cdot (1 - P_{Strong})}$$

- Simplify by assuming the strong monopolist will always signal

$$P_{Strong}^{UPD} = \Pr(M_{Strong} \mid SIG) = \frac{P_{Strong}}{P_{Strong} + \Pr(SIG \mid M_{Weak}) \cdot (1 - P_{Strong})}$$

$$= \frac{P_{Strong}}{P_{Strong} + W_{SIGNAL} \cdot (1 - P_{Strong})}$$

(Just renaming)

Entry deterrence with incomplete information

$$P_{Strong}^{UPD} = \frac{P_{Strong}}{P_{Strong} + W_{SIGNAL} \cdot (1 - P_{Strong})}$$

- Example 1 $\quad W_{SIGNAL} = 0$

$$P_{Strong}^{UPD} = \frac{P_{Strong}}{P_{Strong} + 0 \cdot (1 - P_{Strong})} = 1$$

- If the weak don't signal at all, a signal indicates that the monopolist must be strong

- Example 2 $\quad W_{SIGNAL} = 1$

$$P_{Strong}^{UPD} = \frac{P_{Strong}}{P_{Strong} + 1 \cdot (1 - P_{Strong})} = P_{Strong}$$

- If the weak always signal, then a signal brings no new information.
- The updated probability is equal to the original one (before the signaling)


50000000
RESERVE BANK OF ZIMBABWE
50000000
FIFTY
MILLION
DOLLARS
50000000
AA0358616

=$0.05

Entry deterrence with incomplete information

$$P_{Strong}^{UPD} = \frac{P_{Strong}}{P_{Strong} + W_{SIGNAL} \cdot (1 - P_{Strong})}$$

- Example 3 $\quad P_{Strong} = \frac{1}{2}, W_{SIGNAL} = \frac{1}{2}$

$$P_{Strong}^{UPD} = \frac{\frac{1}{2}}{\frac{1}{2} + \frac{1}{2} \cdot \left(\frac{1}{2}\right)} = \frac{\frac{1}{2}}{\frac{3}{4}} = \frac{2}{3} \approx 0.67 < 0.83$$

- Example 4 $\quad P_{Strong} = \frac{1}{2}, W_{SIGNAL} = \frac{1}{4}$

$$P_{Strong}^{UPD} = \frac{\frac{1}{2}}{\frac{1}{2} + \frac{1}{4} \cdot \left(\frac{1}{2}\right)} = \frac{\frac{1}{2}}{\frac{5}{8}} = \frac{4}{5} = 0.8 < 0.83$$

- Example 5 $\quad P_{Strong} = \frac{1}{2}, W_{SIGNAL} = \frac{1}{6}$

$$P_{Strong}^{UPD} = \frac{\frac{1}{2}}{\frac{1}{2} + \frac{1}{6} \cdot \left(\frac{1}{2}\right)} = \frac{\frac{1}{2}}{\frac{7}{12}} = \frac{6}{7} \approx 0.86 > 0.83$$

Entry deterrence with incomplete information

What is the equilibrium with signaling?

- Strategy of incumbent monopolist:
  - Strong monopolist
    - always sends the signal
    - always fights entry.
  - Weak monopolist
    - sends the signal with some probability if initially $P_{strong} < 5/6$
    - Always concedes if there is entry.

- Strategy of entrant:
  - if doesn't see the signal, then **enters** since then $P_{strong}=0$
  - If the entrant sees the signal, P_{strong} is updated using Bayes' rule
    - the entrant enters if the updated value of $P_{strong} < 5/6$
    - the entrant stays out if the updated value of $P_{strong} > 5/6$

- **5. HIV Tests**.
- Imagine that you are being tested for HIV.
- The test is 98% accurate.
  - If you have HIV, the test shows positive 98% of the time
  - If you do not have HIV, it shows negative 98% of the time.
- 0.05% in the population actually have HIV
- Now your doctor tells you that you tested positive
- What is the probability you have HIV?

$$\boxed{2.4\%}$$

• If you have HIV, the test shows positive 98% of the time	$\Pr[+ \mid HIV] = 0.98$
• If you do not have HIV, it shows negative 98% of the time.	$\Pr[- \mid noHIV] = 0.98$
• 0.05% in the population actually have HIV	$\Pr[+] = 0.05$
• What is the probability you have HIV?	$\Pr[HIV \mid +] = ?$

$$\boxed{\begin{array}{ll} \Pr[+\mid HIV] = 0.98 & \Pr[+] = 0.05 \\ \Pr[-\mid HIV] = 0.98 & \Pr[HIV\mid +] = ? \end{array}}$$

$$\Pr[HIV \mid +] = \frac{\Pr[+\mid HIV]\cdot \Pr[HIV]}{\Pr[+]}$$

$$= \frac{0.98\cdot 0.0005}{0.98\cdot 0.0005 + 0.02\cdot 0.9995} = \frac{0.00049}{0.02048} \approx 0.024$$

$$\Pr[HIV \mid +] = \frac{\Pr[+ \mid HIV] \cdot \Pr[HIV]}{\Pr[+]}$$

$$= \frac{0.98 \cdot 0.0005}{0.98 \cdot 0.0005 + 0.02 \cdot 0.9995} = \frac{0.00049}{0.02048} \approx 0.024$$

- 2.4% <<< 98%.
- Suppose 1,000,000 tests are done:
  - Of these, 500 people have HIV.
    - Of these, 98% test positive on average– 490 people.
  - Of these, 999,500 have no HIV.
    - Of these, 2% test positive on average – 19,990 people.
- Thus 20480 positive tests
  - of which 490 are true positives.
- The probability of having HIV if you test positive is 490/20480
  - about 2.39%.

- This is one reason why HIV testing for the entire population, instead of for high-risk subpopulations, would not be very informative
  - more false positives than true positives.
- What if have a positive result and you go for a second test?
- And you test again positive?

$$\Pr[HIV \mid +] = \frac{\Pr[+ \mid HIV] \cdot \Pr[HIV]}{\Pr[+]} = \frac{0.00049}{0.02048} \approx 0.024$$

$$\Pr[HIV \mid 2^{nd} +] = \frac{\Pr[2^{nd} + \mid HIV] \cdot \Pr[HIV]}{\Pr[2^{nd} +]} =$$

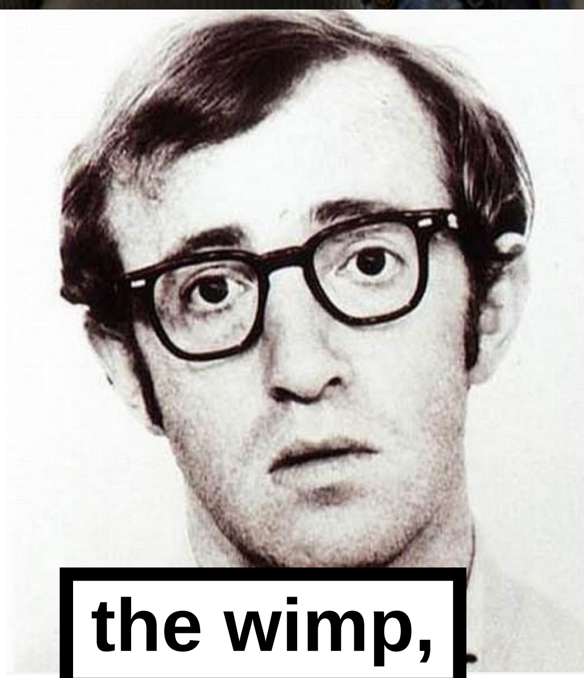
$$= \frac{\Pr[2^{nd} + \mid HIV] \cdot \Pr[HIV]}{\Pr[2^{nd} + \mid HIV] \cdot \Pr[HIV] + \Pr[2^{nd} + \mid \text{no}\, HIV] \cdot \Pr[\text{no}\, HIV]}$$

$$= \frac{0.98 \cdot 0.024}{0.98 \cdot 0.024 + 0.02 \cdot 0.976} \quad = \frac{0.0235}{0.04304} = 0.55$$

- 20 480 positive tests
  - of which 490 are true positives.
  - And 19 990 are negatives
- Going for a test again gives:
  - a total of negative outcomes of
    - 0.98*19 990+0.02*490=19 600
  - Positive outcomes of
    - 0.02*19 990+0.98*490= 880
- We knew there are 490 true positives
- Thus the probability of having HIV is
  - 490/880= 0.557

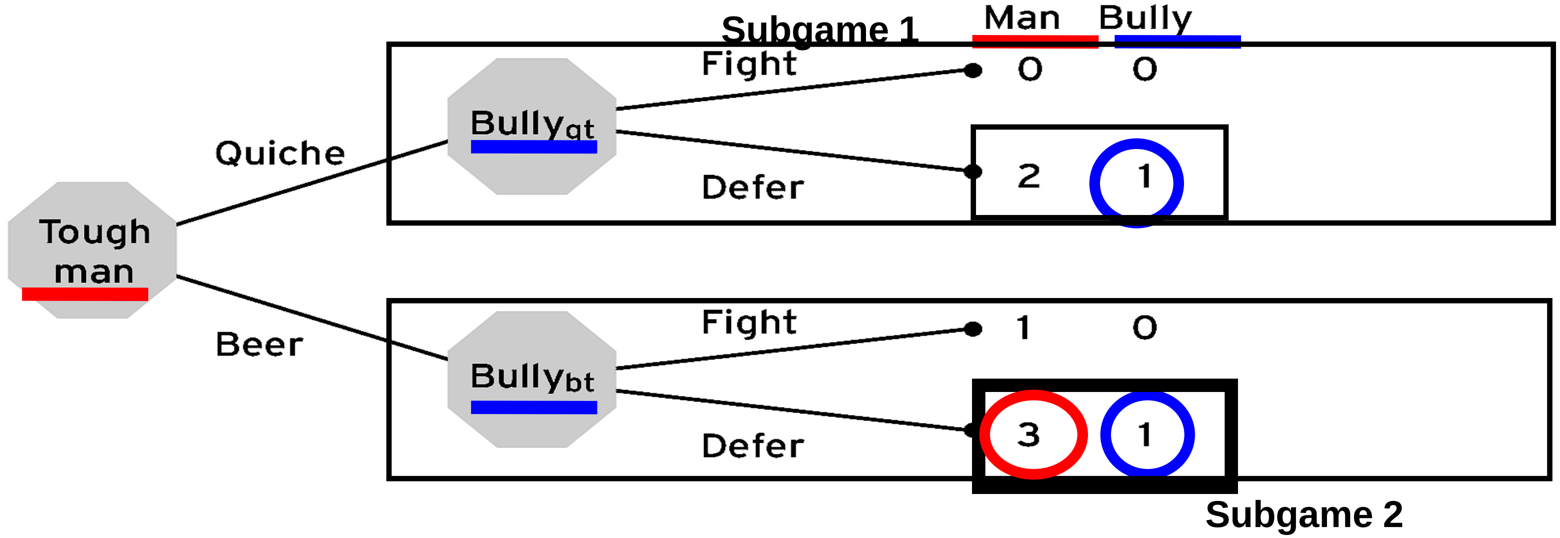
- Look at a case without separating or pooling equilibria in pure strategies
- Must look at mixed strategies

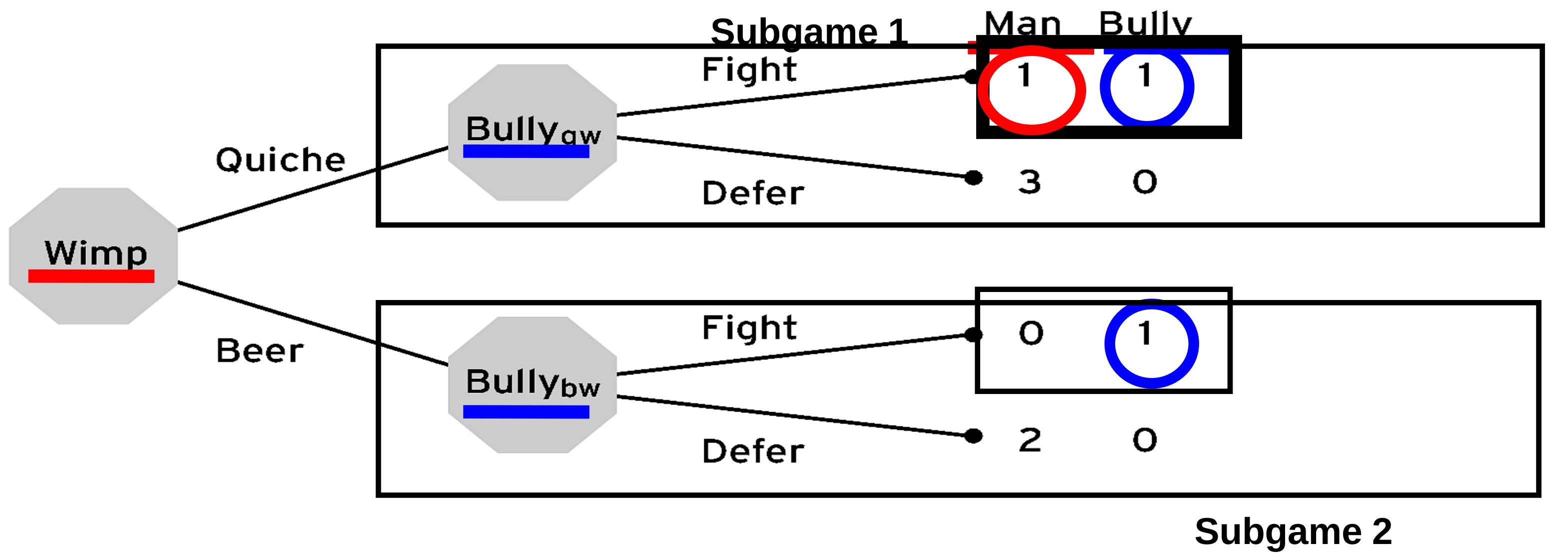

The bully,

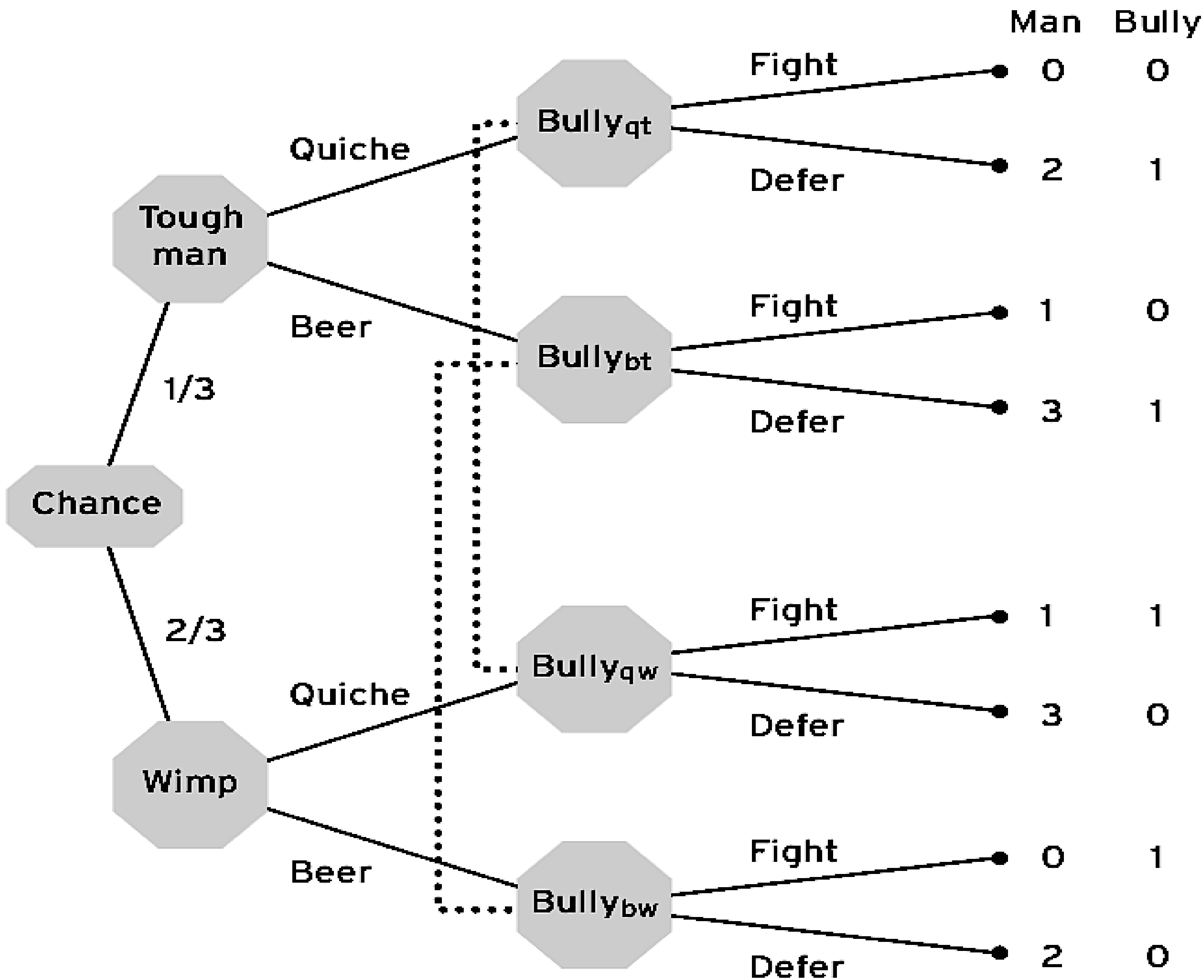

the wimp,


And the tough
man…

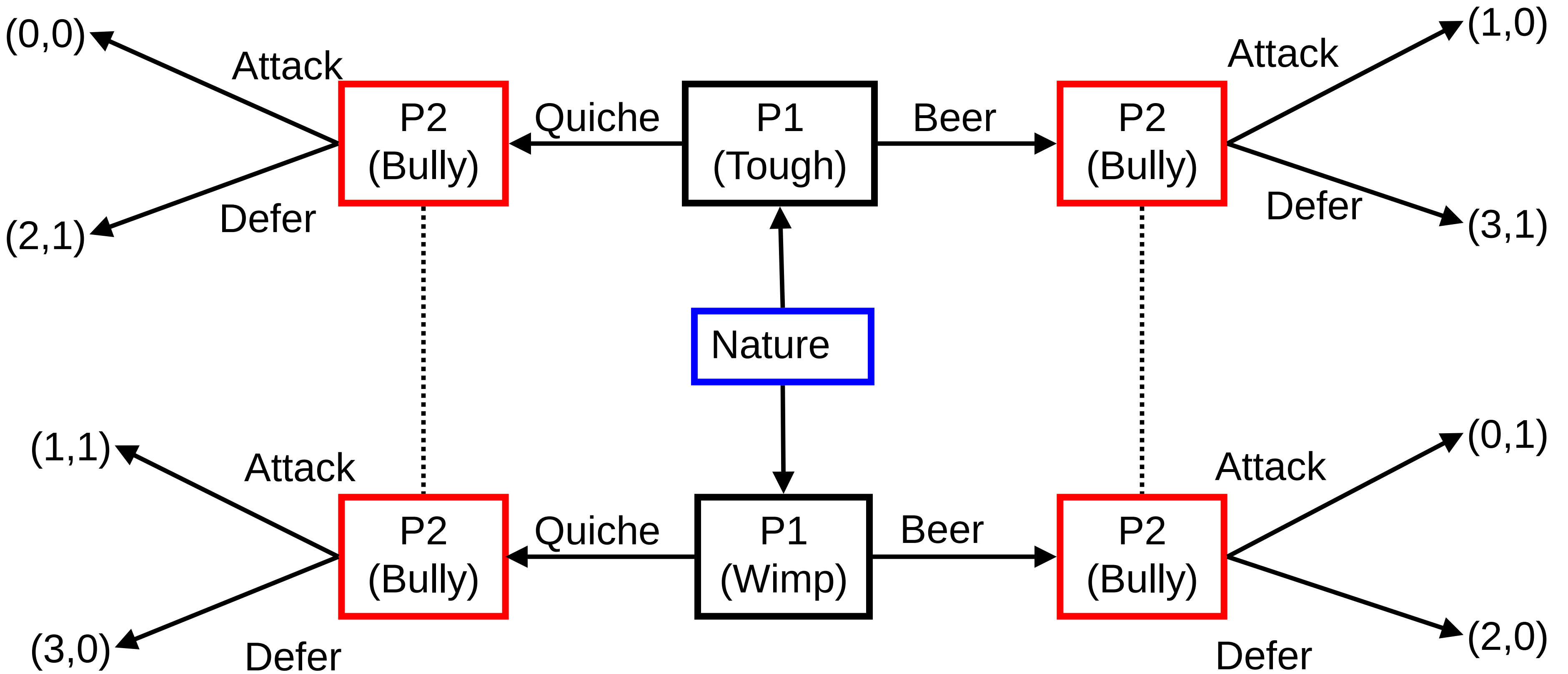
Full information

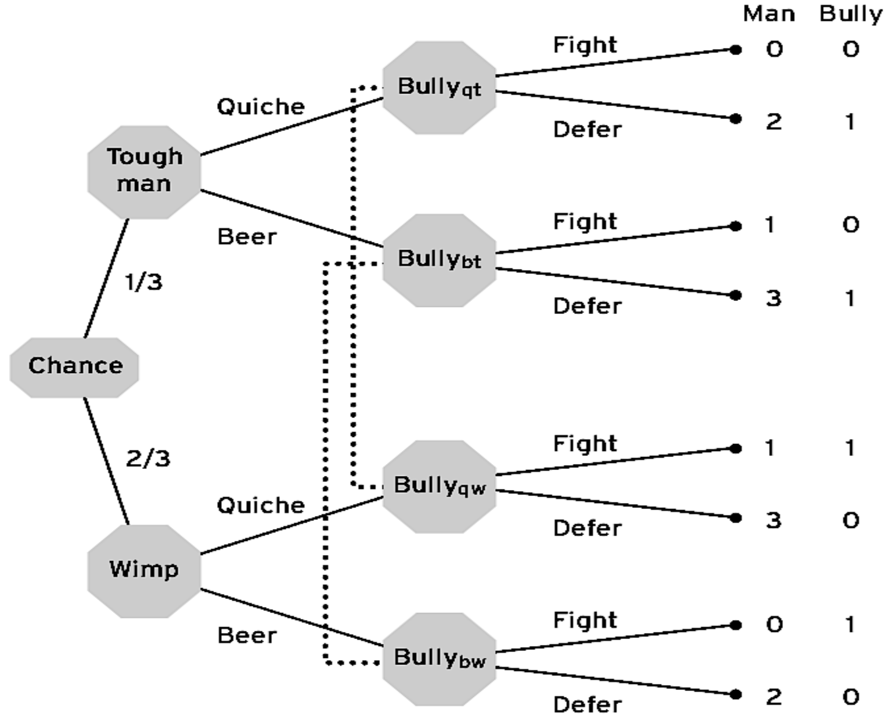


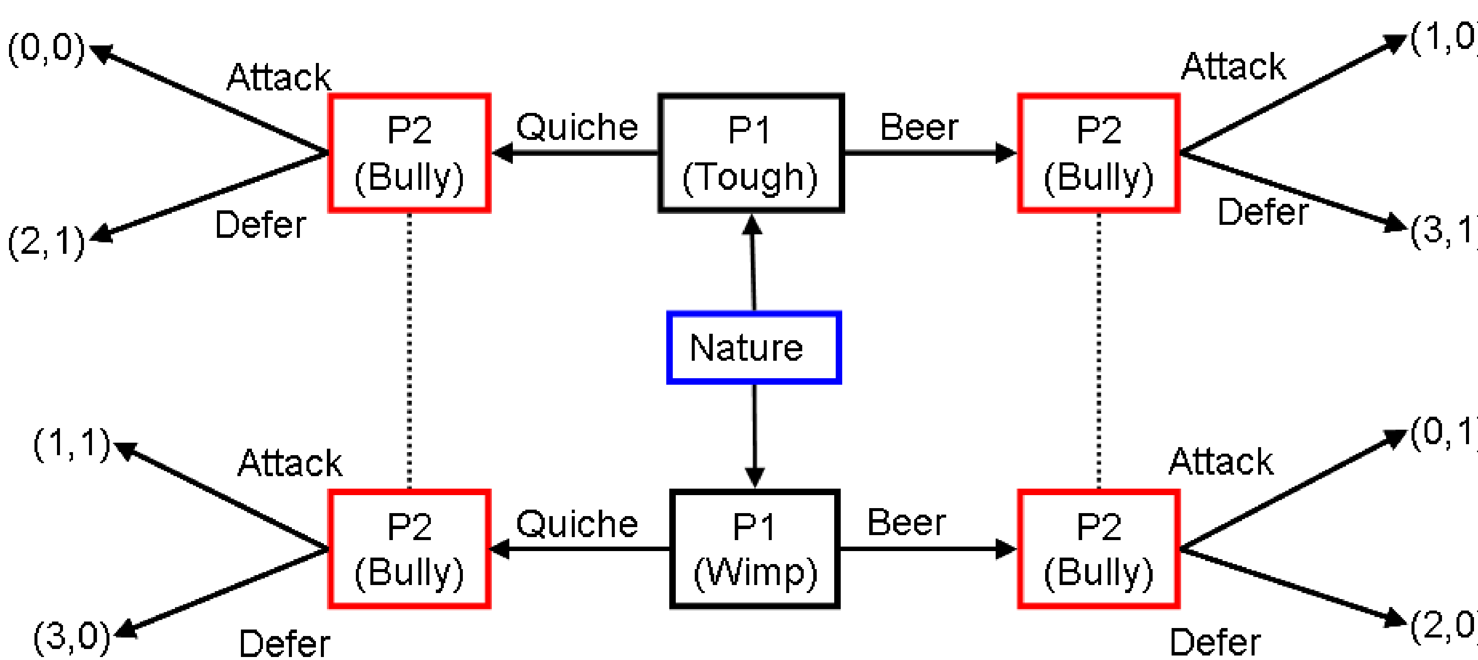


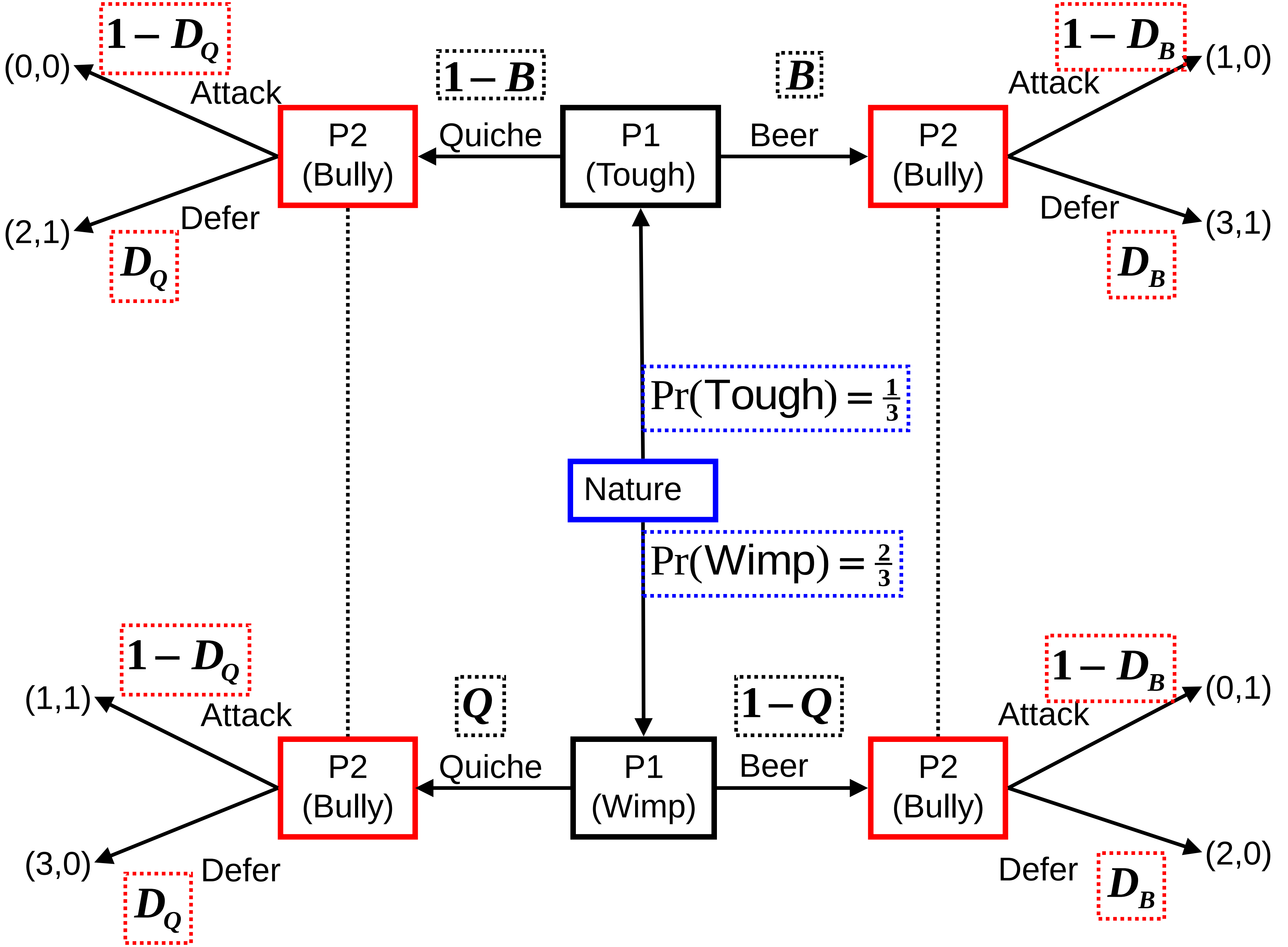


Man
Bully
Chance
1/3
2/3
Tough man
Wimp
Quiche
Beer
Quiche
Beer
Bully_qt
Bully_bt
Bully_qw
Bully_bw
Fight
Defer
Fight
Defer
Fight
Defer
Fight
Defer
0 0
2 1
1 0
3 1
1 1
3 0
0 1
2 0

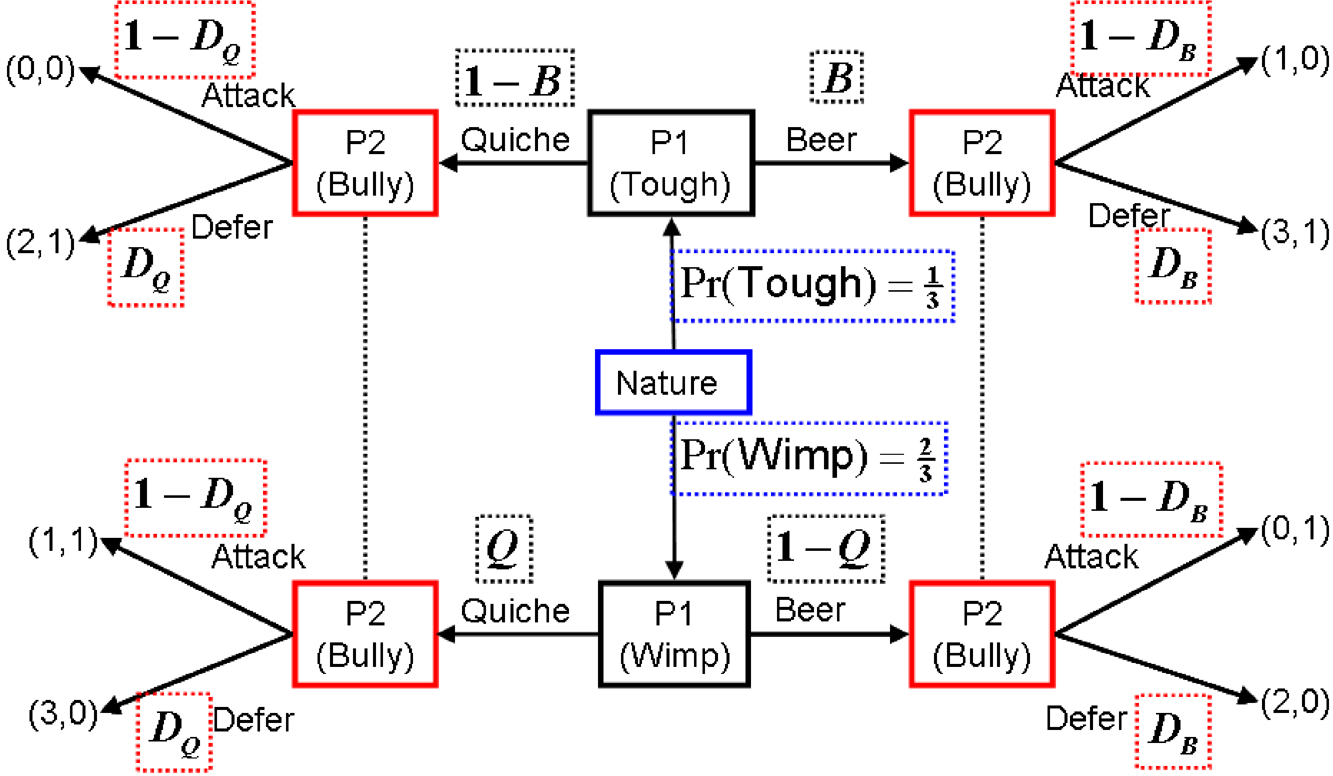

(0,0)
Attack
P2
(Bully)
Quiche
P1
(Tough)
Beer
P2
(Bully)
Attack
(1,0)
(2,1)
Defer
Defer
(3,1)
Nature
(1,1)
Attack
P2
(Bully)
Quiche
P1
(Wimp)
Beer
P2
(Bully)
Attack
(0,1)
(3,0)
Defer
Defer
(2,0)


Man
Bully
Fight
0 0
Bully$_{qt}$
Defer
2 1
Quiche
Tough man
Beer
Fight
1 0
Bully$_{bt}$
Defer
3 1
1/3
Chance
2/3
Fight
1 1
Quiche
Bully$_{qw}$
Defer
3 0
Wimp
Beer
Fight
0 1
Bully$_{bw}$
Defer
2 0

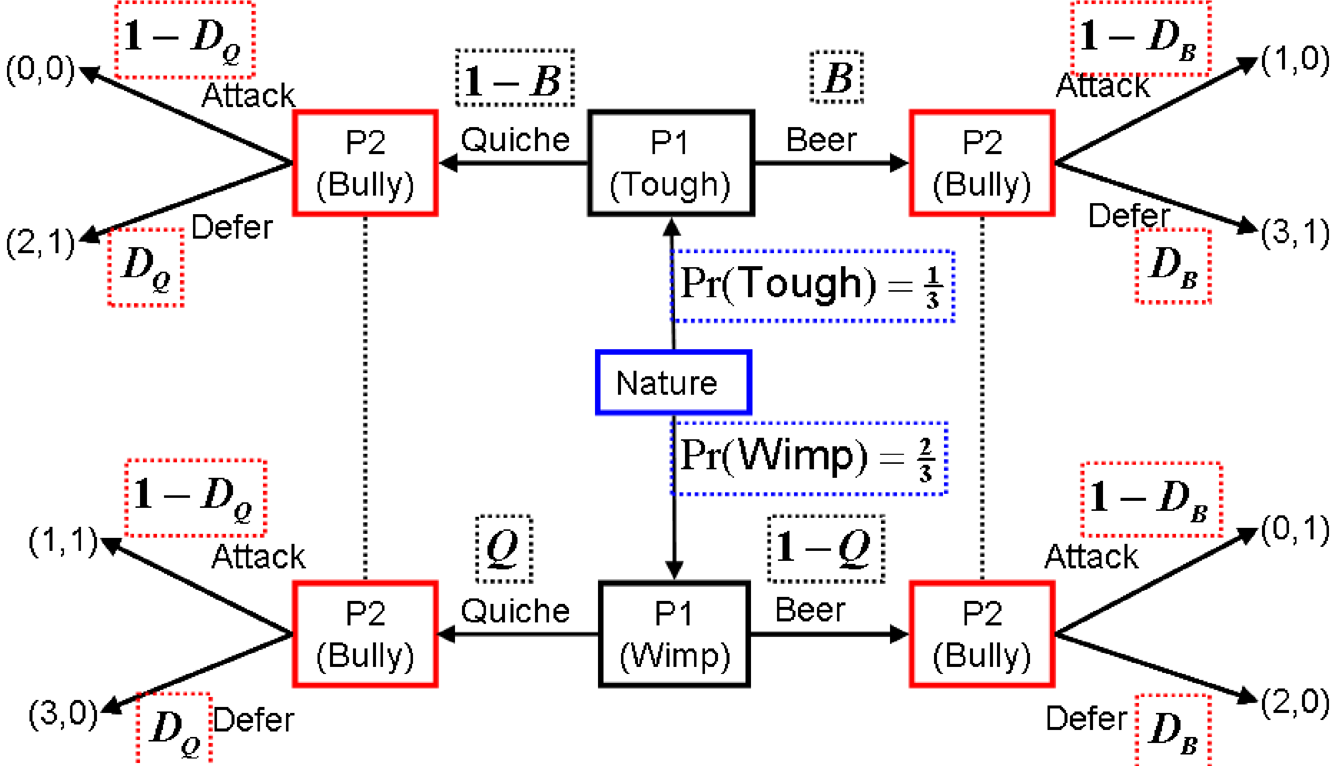

(0,0)
Attack
P2 (Bully)
Defer
(2,1)
Quiche
P1 (Tough)
Beer
P2 (Bully)
Attack
(1,0)
Defer
(3,1)
Nature
(1,1)
Attack
P2 (Bully)
Defer
(3,0)
Quiche
P1 (Wimp)
Beer
P2 (Bully)
Attack
(0,1)
Defer
(2,0)



(0,0)
$1-D_Q$
Attack
$1-B$
B
$1-D_B$
(1,0)
Attack
Quiche
P1
(Tough)
Beer
P2
(Bully)
P2
(Bully)
(2,1)
Defer
D_Q
Defer
(3,1)
D_B
$\Pr(\text{Tough}) = \frac{1}{3}$
Nature
$\Pr(\text{Wimp}) = \frac{2}{3}$
$1-D_Q$
(1,1)
Attack
Q
$1-Q$
$1-D_B$
(0,1)
Attack
P2
(Bully)
Quiche
P1
(Wimp)
Beer
P2
(Bully)
(3,0)
Defer
D_Q
Defer
D_B
(2,0)

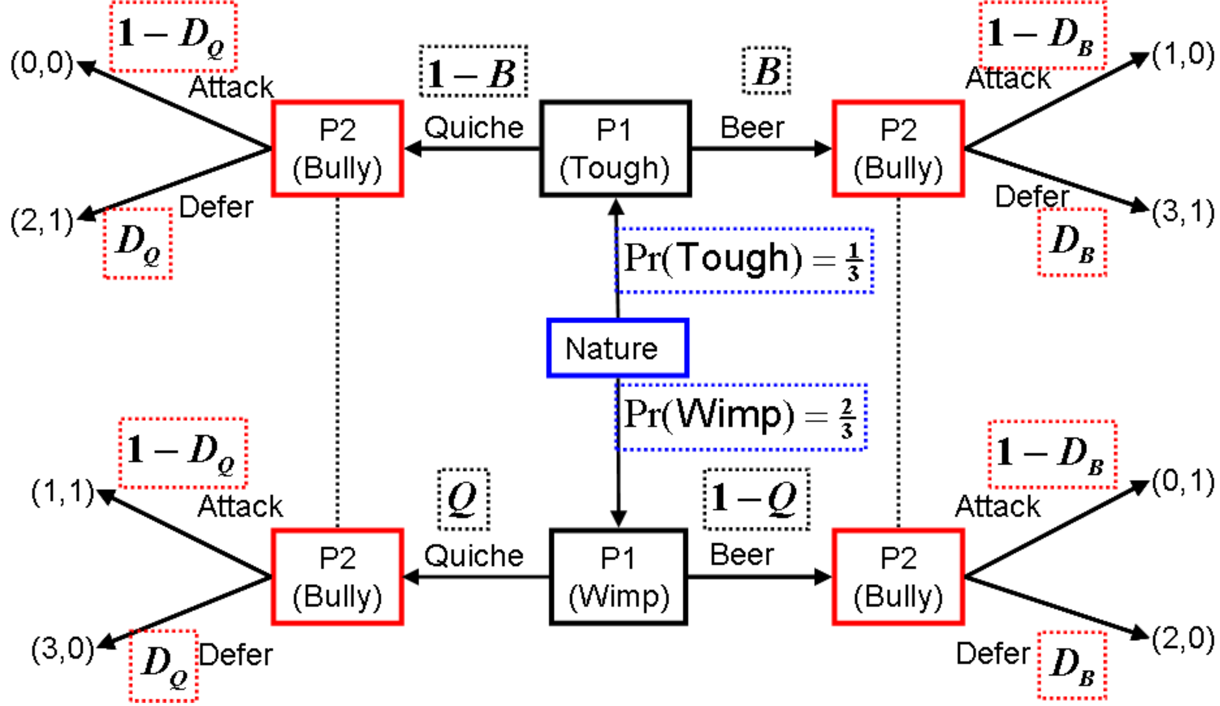


- Separating equilibrium?
  - **P1 separates**
    - **Tough drink beer, Wimps eat quiche**
  - **P2 uses the signal**
    - **Bully: Fight if quiche, not fight if beer**
      - But then wimps deviate by drinking beer
  - Contradiction



- Pooling equilibrium?
  - **P1: doesn't take the signal as informative**
    - **Always fight**
      - But then Tough man always drink beer and Wimp always eat Quiche
      - Is perfect signal!
    - **Always defer?**
      - Bully payoff =⅓
      - But then Tough man will always drink beer and Wimp always eat Quiche
      - Is perfect signal!
- Thus no equilibrium in pure strategies

- Go to end of the game
- Bully either defers (D) or attacks (A)
- What is best for bully:
  - When he sees a man eating Quiche
  - When he sees a man drinking Beer



- Bully sees a man eat Quiche

$$EPO_{B,Fight} = P_{T|Q} \cdot 0 + P_{W|Q} \cdot 1 = P_{W|Q}$$

$$EPO_{B,Defer} = P_{T|Q} \cdot 1 + P_{W|Q} \cdot 0 = P_{T|Q}$$

- Defer only if:

$$EPO_{B,Fight} < EPO_{B,Defer} \quad \leftrightarrow \quad \boxed{P_{W|Q} < P_{T|Q}}$$

How are beliefs $P_{W|Q}$ and $P_{T|Q}$ formed?

- Bully sees a man eat Quiche, he must update his beliefs

Proportion of Tough men

Probability that a Tough man eats Quiche

Probability we have here a Tough man eating Quiche

$$P_{T|Q} = \frac{\frac{1}{3}(1-B)}{\frac{1}{3}(1-B)+\frac{2}{3}Q}$$

Probability we have here a Tough man eating Quiche

Probability that a Quiche is eaten by a Wimp

Total Probability that a Quiche is eaten

- Bully sees a man eat Quiche, he must update his beliefs

Proportion of Wimps

Probability that a Wimp eats Quiche

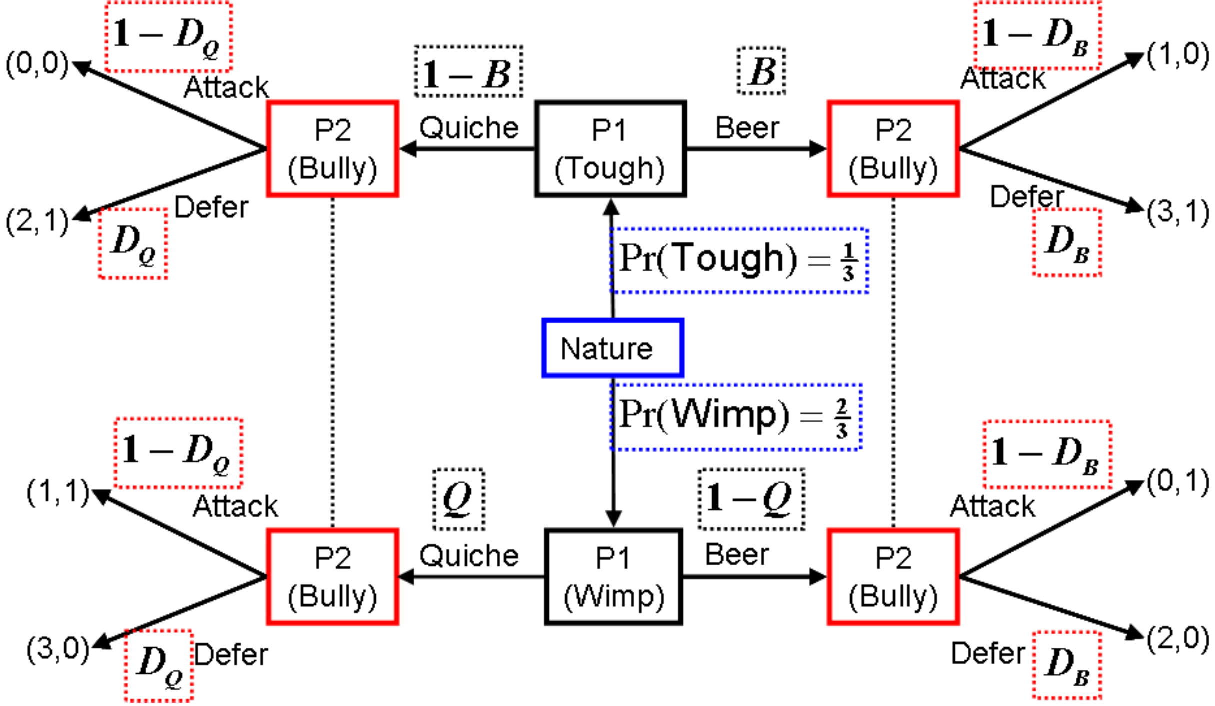
Probability we have here a Wimp eating Quiche

$$P_{W|Q} = \frac{\frac{2}{3}Q}{\frac{1}{3}(1-B)+\frac{2}{3}Q}$$

Probability we have here a Tough man eating Quiche

Probability that a Quiche is eaten by a Wimp

Total Probability that a Quiche is eaten



- Bully sees a man eat Quiche

$$EPO_{B,Fight} = P_{T|Q} \cdot 0 + P_{W|Q} \cdot 1 = P_{W|Q}$$

$$EPO_{B,Defer} = P_{T|Q} \cdot 1 + P_{W|Q} \cdot 0 = P_{T|Q}$$

- Defer only if:

$$EPO_{B,Fight} < EPO_{B,Defer} \leftrightarrow \boxed{P_{W|Q} < P_{T|Q}}$$

- Defer only if:

$$\boxed{P_{W|Q} < P_{T|Q}}$$

$$P_{W|Q} = \frac{\frac{2}{3}Q}{\frac{1}{3}(1-B) + \frac{2}{3}Q}$$

$$P_{T|Q} = \frac{\frac{1}{3}(1-B)}{\frac{1}{3}(1-B) + \frac{2}{3}Q}$$

$$P_{W|Q} < P_{T|Q}$$

$$\frac{\frac{2}{3}Q}{\frac{1}{3}(1-B) + \frac{2}{3}Q} < \frac{\frac{1}{3}(1-B)}{\frac{1}{3}(1-B) + \frac{2}{3}Q}$$

$$\frac{2}{3}Q < \frac{1}{3}(1-B)$$

$$2Q < (1-B)$$

1. The bully defers at Quiche if: $2Q < (1 - B)$.
2. The bully fights at Quiche if: $2Q > (1 - B)$.
3. The bully randomizes at Quiche if: $2Q = (1 - B)$.

- Bully sees a man drink Beer

$$EPO_{B,Fight} = P_{T|B} \cdot 0 + P_{W|B} \cdot 1 = P_{W|B}$$

$$EPO_{B,Defer} = P_{T|B} \cdot 1 + P_{W|B} \cdot 0 = P_{T|B}$$

- Defer only if:

$$EPO_{B,Fight} < EPO_{B,Defer} \leftrightarrow \boxed{P_{W|B} < P_{T|B}}$$

How are beliefs $P_{W|B}$ and $P_{T|B}$ formed?

- Bully sees a man drink Beer, he must update his beliefs

Proportion of Tough men

Probability that a Tough man drinks Beer

Probability we have here a Tough man drinking Beer

$$P_{T|B} = \frac{\frac{1}{3}B}{\frac{1}{3}B + \frac{2}{3}(1-Q)}$$

Probability we have here a Tough man drinking Beer

Probability that a Beer is drunk by a Wimp

Total Probability that a Beer is drunk

- Bully sees a man drink Beer, he must update his beliefs

Proportion of Wimps

Probability that a Wimp drinks Beer

Probability we have here a Wimp drinking Beer

$$P_{W|B} = \frac{\overbrace{\tfrac{2}{3}(1-Q)}}{\tfrac{1}{3}B + \tfrac{2}{3}(1-Q)}$$

Probability we have here a Tough man drinking Beer

Probability that a Beer is drunk by a Wimp

Total Probability that a Beer is drunk

- Bully sees a man drink Beer

$$EPO_{B,Fight} = P_{T|B} \cdot 0 + P_{W|B} \cdot 1 = P_{W|B}$$

$$EPO_{B,Defer} = P_{T|B} \cdot 1 + P_{W|B} \cdot 0 = P_{T|B}$$

- Defer only if:

$$EPO_{B,Fight} < EPO_{B,Defer} \leftrightarrow \boxed{P_{W|B} < P_{T|B}}$$

- Bully sees a man drink Beer, he must update his belief

- Defer only if:

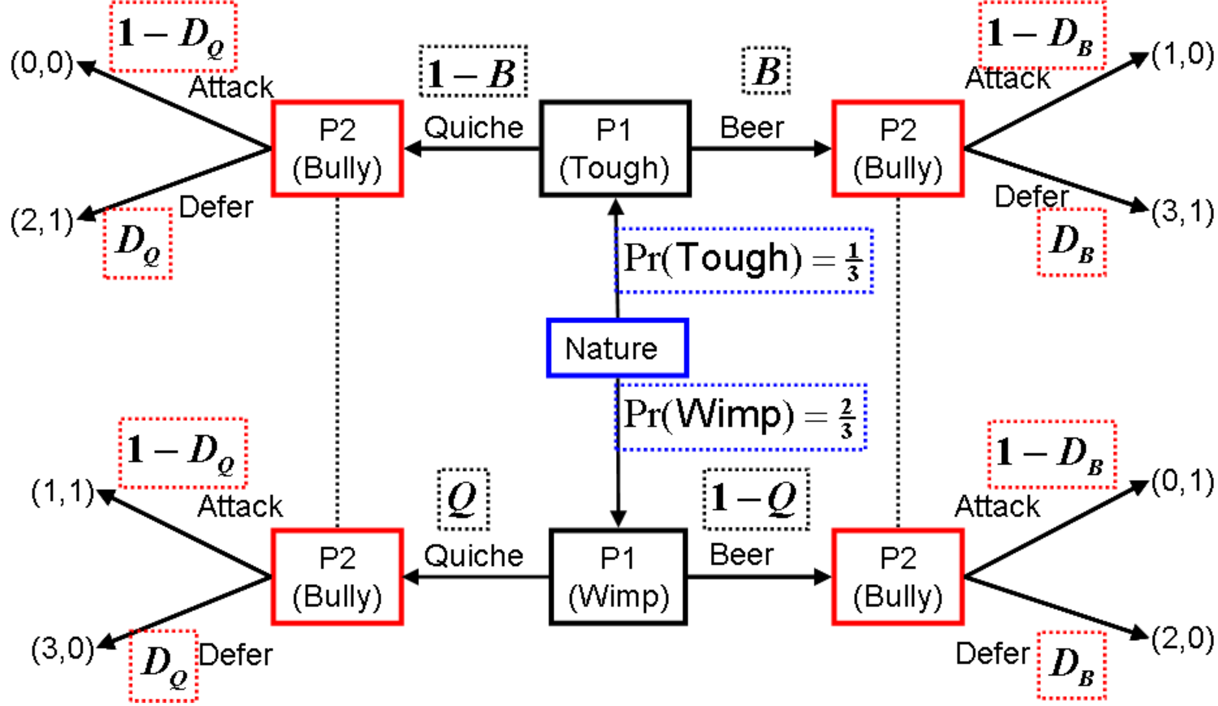
$$\left.\begin{aligned}&\boxed{P_{W|B} < P_{T|B}}\\&P_{W|B} = \frac{\frac{2}{3}(1-Q)}{\frac{1}{3}B + \frac{2}{3}(1-Q)}\\&P_{T|B} = \frac{\frac{1}{3}B}{\frac{1}{3}B + \frac{2}{3}(1-Q)}\end{aligned}\right\}$$

$$P_{W|B} < P_{T|B}$$

$$\frac{\frac{2}{3}(1-Q)}{\frac{1}{3}B + \frac{2}{3}(1-Q)} < \frac{\frac{1}{3}B}{\frac{1}{3}B + \frac{2}{3}(1-Q)}$$

$$\frac{2}{3}(1-Q) < \frac{1}{3}B$$

$$2(1-Q) < B$$



a) The bully defers at Quiche if:
2Q < (1 – B).

b) The bully fights at Quiche if:
2Q > (1 – B).

c) The bully randomises at Quiche if:
2Q = (1 – B).

d) The bully defers at beer if
B > 2(1 – Q).

e) The bully fights at beer if
B < 2(1 – Q).

f) The bully randomises at beer if
B = 2(1 – Q).

a) The bully defers at Quiche if: 2Q < (1 – B).

b) The bully fights at Quiche if: 2Q > (1 – B).

c) ~~The bully randomises at Quiche if: 2Q = (1 – B).~~

d) The bully defers at beer if: B > 2(1 – Q).

e) The bully fights at beer if: B < 2(1 – Q).

f) The bully randomises at beer if: B = 2(1 – Q).

c) The bully randomises at Quiche if: 2Q = (1 – B).

- If true, then B = 1 – 2Q
  - but then B < 2(1 – Q) e)
  - Thus the bully fights at Beer
- Now the Wimp will thus always eat Quiche: Q=1
  - But then in c): 2 = 1 – B: B=-1??? CONTR!
  - c) cannot be true

a) The bully defers at Quiche if: $2Q < (1 - B)$.

b) The bully fights at Quiche if: $2Q > (1 - B)$.

c) ~~The bully randomises at Quiche if: $2Q = (1 - B)$.~~

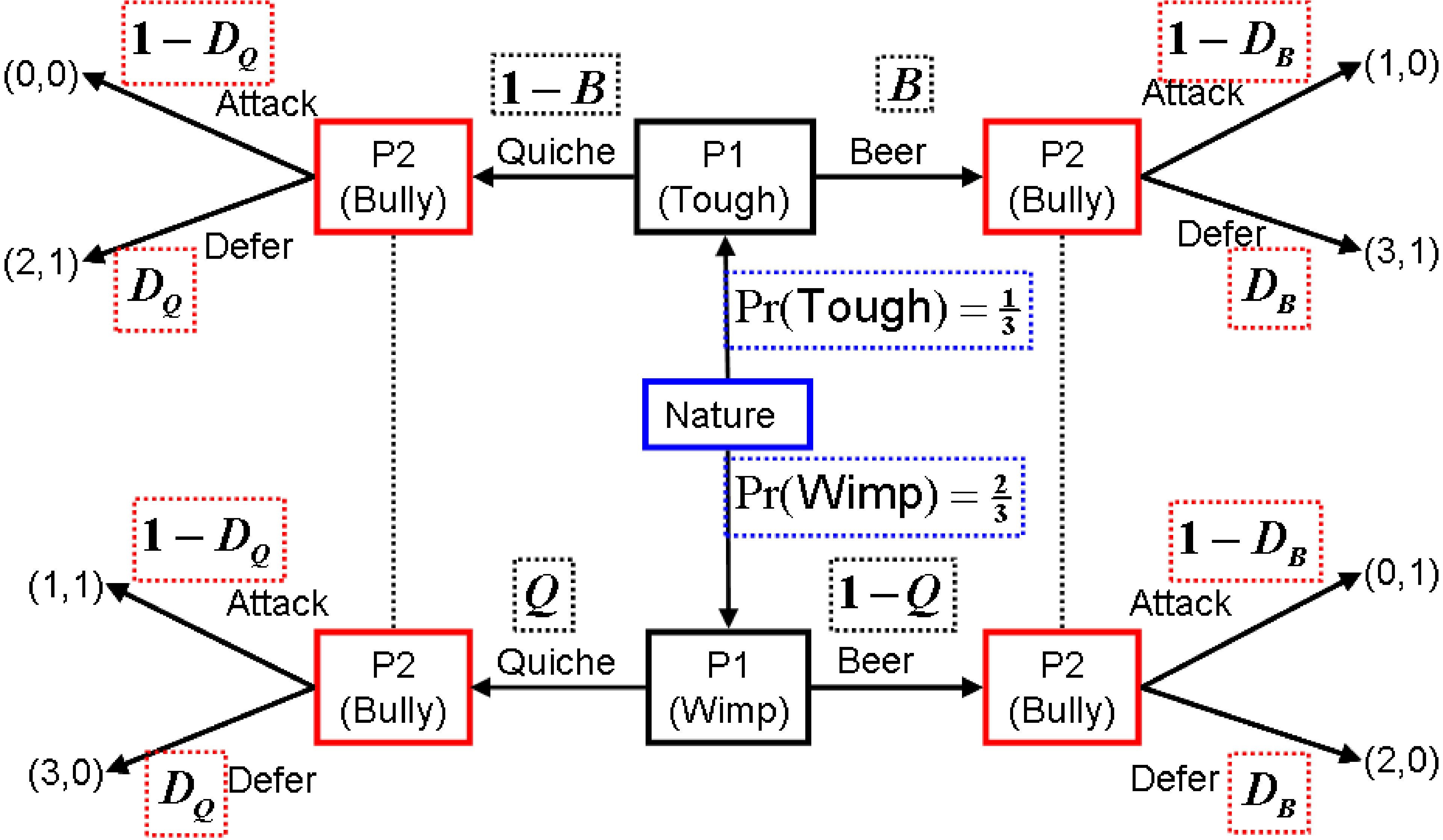
d) The bully defers at beer if: $B > 2(1 - Q)$.

e) The bully fights at beer if: $B < 2(1 - Q)$.

f) The bully randomises at beer if: $B = 2(1 - Q)$.

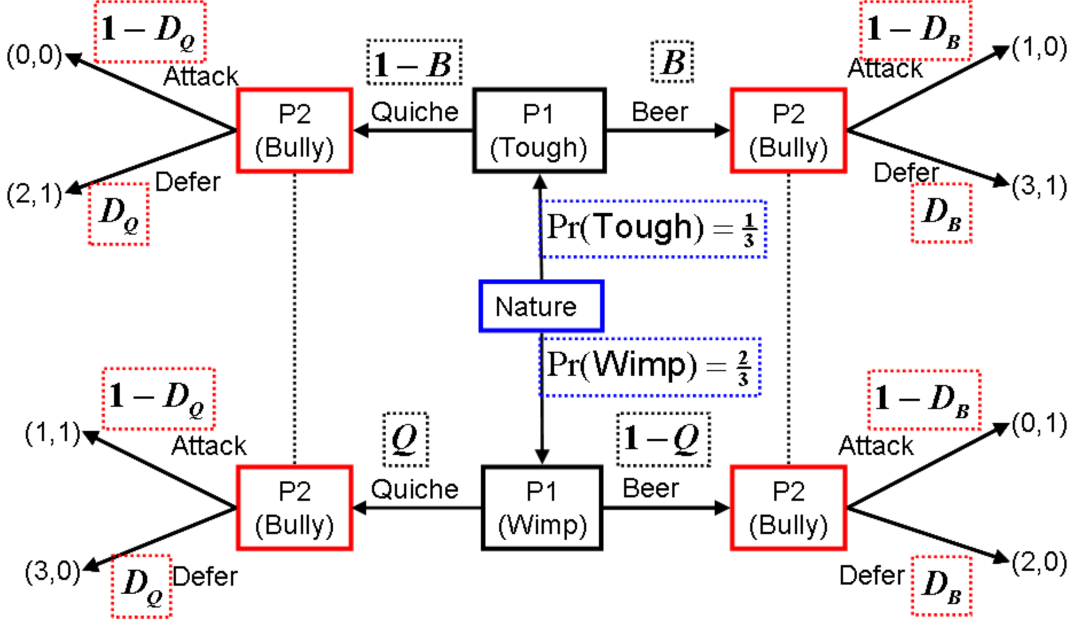
f) The bully randomises at Beer if: $B = 2(1 - Q)$.

- If true, then $2Q = 2 - B$
  - but then $2Q > 1 - B$ b)
  - Thus the **bully always fights at Quiche**
- Now the Tough man will always drink Beer: $B=1$
  - But then in f): $1 = 2(1 - Q) \to Q = \frac{1}{2}$
    - Thus Wimp randomises: drinks beer with probability ½
    - What is the payoff for the Wimp?



Nature
Pr(Tough) = 1/3
Pr(Wimp) = 2/3
P1
(Tough)
P1
(Wimp)
Quiche
Beer
1 − B
B
Q
1 − Q
P2
(Bully)
Attack
Defer
1 − D_Q
D_Q
1 − D_B
D_B
(0,0)
(2,1)
(1,0)
(3,1)
(1,1)
(3,0)
(0,1)
(2,0)

- The Wimp randomizes with prob ½ between Beer and Quiche
- The Bully randomizes at Beer
- The Bully always fights at Quiche



$$EPO_{Wimp,Quiche} = 1$$

$$EPO_{Wimp,Beer} = 2D_B + 0(1-D_B) = 2D_B$$

To keep the Wimp randomizing with prob. ½, the exp. Payoffs must be equal. Thus

$$\tfrac{1}{2} \cdot EPO_{Wimp,Quiche} = \tfrac{1}{2} EPO_{Wimp,Beer}$$

$$EPO_{Wimp,Quiche} = EPO_{Wimp,Beer}$$

$$1 = 2D_B$$

$$D_B = \tfrac{1}{2}$$

- The man’s strategy
  - The tough man always drinks beer
  - The wimp eats quiche with probability ½ and drinks beer with probability ½.
- The bully’s strategy
  - If the man eats quiche; fight
  - If the man drinks beer; fight with probability ½ and defer with probability ½

EVROPSKÁ UNIE
Evropské strukturální a investiční fondy
Operační program Výzkum, vývoj a vzdělávání

MINISTERSTVO ŠKOLSTVÍ, MLÁDEŽE A TĚLOVÝCHOVY

Národohospodářská fakulta VŠE v Praze