# Aproximace funkce

## Metoda nejmenších čtverců

## a

## Taylorův polynom

EVROPSKÁ UNIE
Evropské strukturální a investiční fondy
Operační program Výzkum, vývoj a vzdělávání

MINISTERSTVO ŠKOLSTVÍ, MLÁDEŽE A TĚLOVÝCHOVY

# Aritmetický vektorový prostor

$$\mathbf{R}^n = \{(x_1, x_2, \cdots, x_n), x_i \in \mathbf{R}, i = 1, \cdots, n\}$$

$$\mathbf{x} = (x_1, x_2, \cdots, x_n), \mathbf{y} = (y_1, y_2, \cdots, y_n),$$

Součet a násobek:

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n)$$

$$k\mathbf{x} = (kx_1, kx_2, \cdots, kx_n)$$

# Skalární součin v $\mathbf{R}^n$

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

Platí:

$\mathbf{x}\,\mathbf{y} = \mathbf{y}\,\mathbf{x}$

$\mathbf{x} \cdot (\alpha\,\mathbf{y} + \beta\,\mathbf{z}) = \alpha\,\beta(\mathbf{x}\,\mathbf{y} + \mathbf{x}\,\mathbf{z})$

$\mathbf{x}\,\mathbf{x} \geq 0, \mathbf{x}\,\mathbf{x} = 0 \Leftrightarrow \mathbf{x} = \mathbf{o}$

Norma vektoru: $\|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$

$\mathbf{x} \cdot \mathbf{y} = 0 \Leftrightarrow \mathbf{x} = \mathbf{o} \;\vee\; \mathbf{x} \perp \mathbf{y}$

# Lineární regrese

Dáno: $[x_1, y_1], [x_2, y_2], \cdots, [x_n, y_n]$

Hledáme: $f: Y = kX + q$

Aby pro $Y_1 = f(x_1), Y_2 = f(x_2), \cdots, Y_n = f(x_n)$

a $y_1, y_2, \cdots, y_n$

Bylo

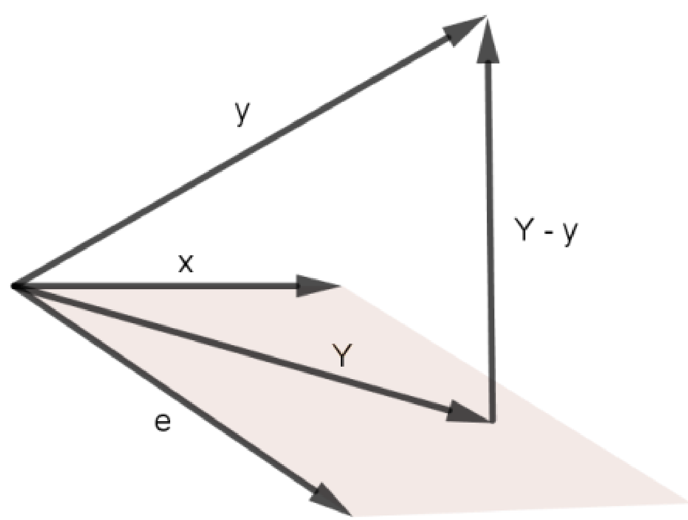
$S = (y_1 - Y_1)^2 + (y_2 - Y_2)^2 + \cdots + (y_n - Y_n)^2$ minimální.

Po dosazení:

$$S = (y_1 - kx_1 - q)^2 + (y_2 - kx_2 - q)^2 + \cdots + (y_n - kx_n - q)^2$$

$$S = (y_1 - kx_1 - q)^2 + (y_2 - kx_2 - q)^2 + \cdots + (y_n - kx_n - q)^2$$

Označíme: $\mathbf{y} = (y_1, y_2, \cdots, y_n)$, $\mathbf{x} = (x_1, x_2, \cdots, x_n)$, $\mathbf{e} = (1,1,\cdots,1)$

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$S = \left(\mathbf{y} - (k\mathbf{x} + \mathrm{q}\mathbf{e})\right)^2$ bude min. $\Leftrightarrow$

$$(\mathbf{y} - (k\mathbf{x} + \mathrm{q}\mathbf{e})) \cdot \mathbf{x} = 0,$$
$$(\mathbf{y} - (k\mathbf{x} + \mathrm{q}\mathbf{e})) \cdot \mathbf{e} = 0$$

$$k(\mathbf{x} \cdot \mathbf{x}) + q(\mathbf{x} \cdot \mathbf{e}) = \mathbf{x} \cdot \mathbf{y},$$
$$k(\mathbf{x} \cdot \mathbf{e}) + q(\mathbf{e} \cdot \mathbf{e}) = \mathbf{e} \cdot \mathbf{y}.$$

# Lineární regrese

Soustava normálních rovnic pro neznámé *k,q*

$$k(\mathbf{x}\cdot\mathbf{x}) + q(\mathbf{x}\cdot\mathbf{e}) = \mathbf{x}\cdot\mathbf{y},$$
$$k(\mathbf{x}\cdot\mathbf{e}) + q(\mathbf{e}\cdot\mathbf{e}) = \mathbf{e}\cdot\mathbf{y}.$$

Cramerovo pravidlo:

$$k = \frac{\begin{vmatrix}\mathbf{xy} & \mathbf{xe}\\ \mathbf{ey} & \mathbf{ee}\end{vmatrix}}{\begin{vmatrix}\mathbf{xx} & \mathbf{xe}\\ \mathbf{xe} & \mathbf{ee}\end{vmatrix}}, \qquad q = \frac{\begin{vmatrix}\mathbf{xx} & \mathbf{x}\boldsymbol{y}\\ \mathbf{ex} & \mathbf{e}y\end{vmatrix}}{\begin{vmatrix}\mathbf{xx} & \mathbf{xe}\\ \mathbf{xe} & \mathbf{ee}\end{vmatrix}}$$

# Lineární regrese

$[-1,1], [0,2], [2,1], [4,3]$

$\mathbf{x} = (-1,0,2,4), \mathbf{y} = (1,2,1,3)$

$\mathbf{e} = (1,1,1,1)$

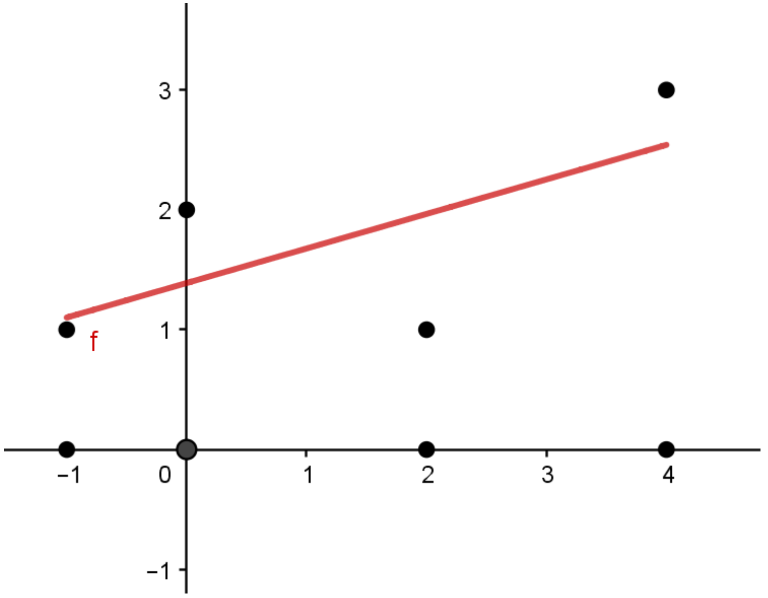
$\mathbf{xx} = 21, \mathbf{xe} = 5, \mathbf{ee} = 4, \mathbf{xy} = 13,$
$\mathbf{ye} = 7$

$$\begin{pmatrix} 21 & 5 \\ 5 & 4 \end{pmatrix} \begin{pmatrix} k \\ q \end{pmatrix} = \begin{pmatrix} 13 \\ 7 \end{pmatrix}$$

$$\begin{pmatrix} k \\ q \end{pmatrix} = \frac{1}{59} \begin{pmatrix} 4 & -5 \\ -5 & 21 \end{pmatrix} \begin{pmatrix} 13 \\ 7 \end{pmatrix}$$

$$\begin{pmatrix} k \\ q \end{pmatrix} = \frac{1}{59} \begin{pmatrix} 17 \\ 82 \end{pmatrix}$$

$$Y = \frac{17}{59} X + \frac{82}{59}$$

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# Metoda nejmenších čtverců

Dáno: $\mathbf{u} \in \mathbf{R}^n$

Hledáme: $\mathbf{v} = v_1\mathbf{b}_1 + v_2\mathbf{b}_2 + \cdots + v_m\mathbf{b}_m$

vzhledem k bázi $\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_m \in \mathbf{R}^n, m \leq n$

tak, aby norma $\|\mathbf{u} - \mathbf{v}\|^2$ byla minimální, tj.

$\mathbf{u} - \mathbf{v}$ musí být kolmý na vektory $\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_m$ , tj.

$$(\mathbf{u} - \mathbf{v}) \cdot \mathbf{b}_i = 0, i = 1,2, \cdots, m,$$

$$\mathbf{u} \cdot \mathbf{b}_i = (v_1\mathbf{b}_1 + v_2\mathbf{b}_2 + \cdots + v_m\mathbf{b}_m) \cdot \mathbf{b}_i$$

# Soustava normálních rovnic

$$(v_1\mathbf{b}_1 + v_2\mathbf{b}_2 + \cdots + v_m\mathbf{b}_m) \cdot \mathbf{b}_i = \mathbf{u} \cdot \mathbf{b}_i,$$

$$\begin{pmatrix} \mathbf{b}_1\mathbf{b}_1 & \mathbf{b}_1\mathbf{b}_2 & \cdots & \mathbf{b}_1\mathbf{b}_m \\ \mathbf{b}_2\mathbf{b}_1 & \mathbf{b}_2\mathbf{b}_2 & \cdots & \mathbf{b}_2\mathbf{b}_m \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{b}_m\mathbf{b}_1 & \mathbf{b}_m\mathbf{b}_2 & \cdots & \mathbf{b}_m\mathbf{b}_m \end{pmatrix} \begin{pmatrix} v_1 \\ v_1 \\ \cdots \\ v_m \end{pmatrix} = \begin{pmatrix} \mathbf{u}\mathbf{b}_1 \\ \mathbf{u}\mathbf{b}_2 \\ \cdots \\ \mathbf{u}\mathbf{b}_m \end{pmatrix}.$$

# Kvadratická regrese

$[-1,1], [0,2], [2,1], [4,3]$ $\quad Y = -0{,}325\,X^2 + 1{,}546\,X + 1{,}525$

$Y = aX^2 + bX + c$

$\mathbf{u} = (1,2,1,3)$

$\mathbf{b}_1 = (1,0,4,16)$

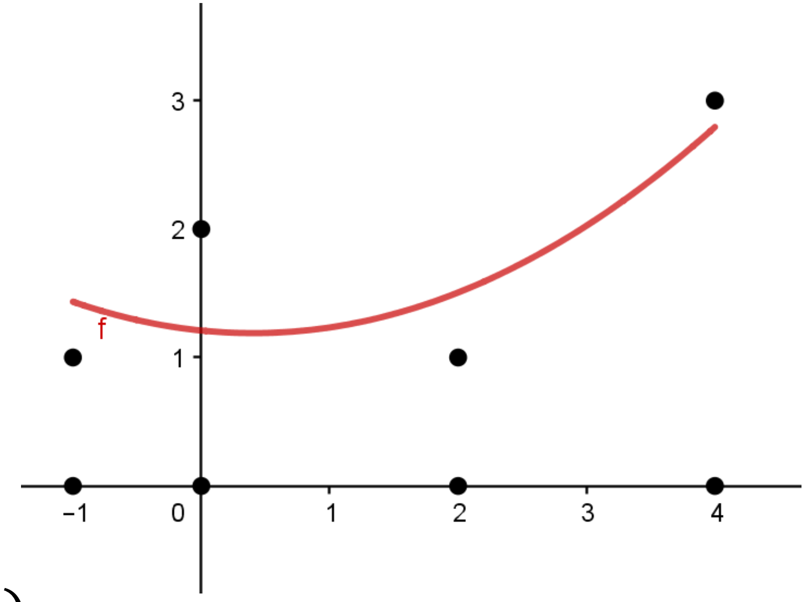
$\mathbf{b}_2 = (-1,0,2,4)$

$\mathbf{b}_3 = (1,1,1,1)$

$\mathbf{v} = a\,\mathbf{b}_1 + b\,\mathbf{b}_2 + c\,\mathbf{b}_3$

$$\begin{pmatrix} 273 & 71 & 21 \\ 71 & 21 & 5 \\ 21 & 5 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 53 \\ 13 \\ 7 \end{pmatrix}$$

$(a, b, c) = (-0{,}325; 0{,}438; -1{,}166)$

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# Soustava normálních rovnic

$$\begin{pmatrix} 1 & 0 & 4 & 16 \\ -1 & 0 & 2 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 4 & 2 & 1 \\ 16 & 4 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 & 0 & 4 & 16 \\ -1 & 0 & 2 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 273 & 71 & 21 \\ 71 & 21 & 5 \\ 21 & 5 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 53 \\ 13 \\ 7 \end{pmatrix}$$

# Aproximace fce Taylorovým polynomem

$$T_n(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots + a_n(x-c)^n$$

$$f^{(n)}(c) = T_n^{(n)}(c)$$

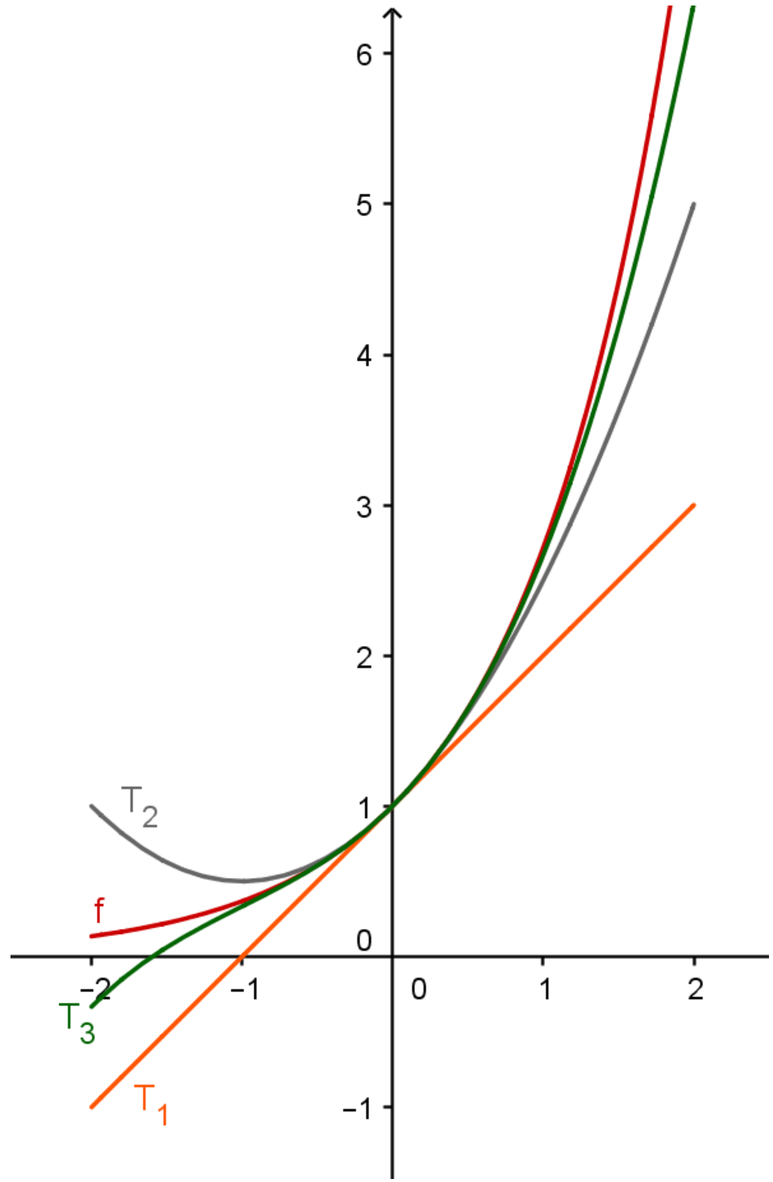
$$T_n(x) = f(c) + f'(c)(x-c) + \frac{1}{2}f''(c)(x-c)^2 + \cdots + \frac{1}{n!}f^{(n)}(c)(x-c)^n$$

Lagrangeův tvar zbytku po n-tém členu:

$$R_{n+1}(x) = \frac{1}{(n+1)!}f^{(n+1)}(\tau)(x-c)^{n+1}, \tau \in U(c)$$

$$c = 0, f(x) = e^x$$

$$T_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$$

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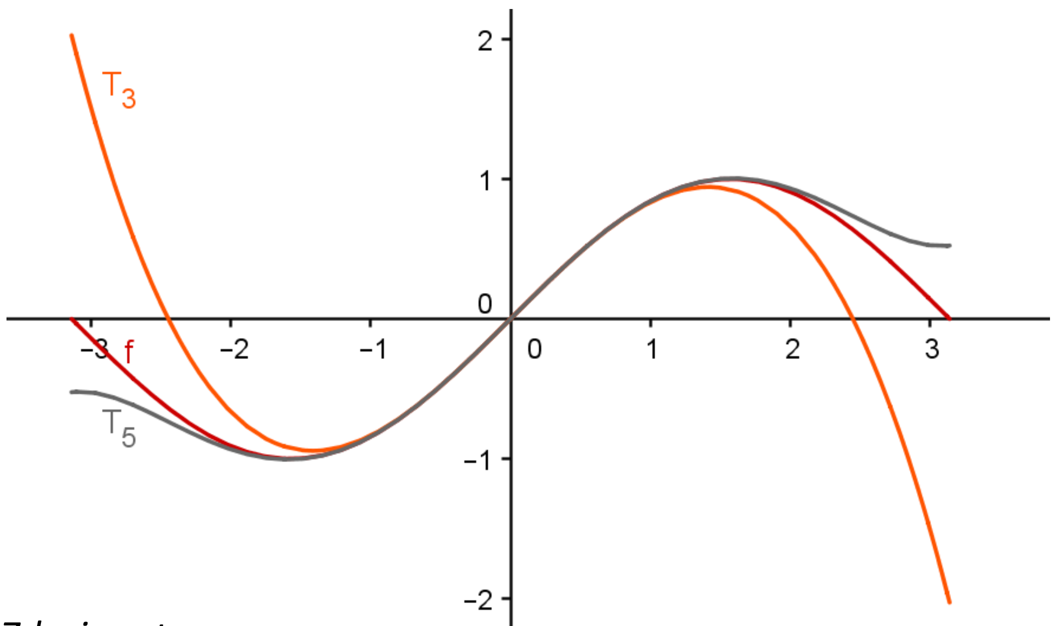
$$C = 0, \; f(x) = \sin x$$

$$T_n(x) = \sum_{k=1}^{n} (-1)^{k-1} \frac{x^{2k-1}}{(2k-1)!}$$

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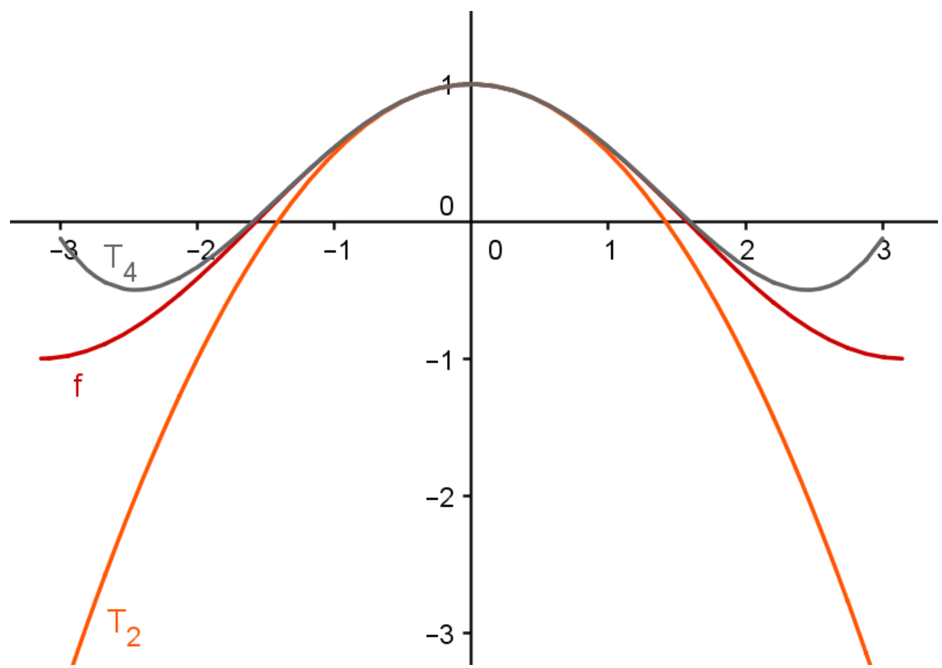
$$C = 0,\ f(x) = \cos x$$

$$T_n(x) = \sum_{k=0}^{n} (-1)^k \frac{x^{2k}}{(2k)!}$$

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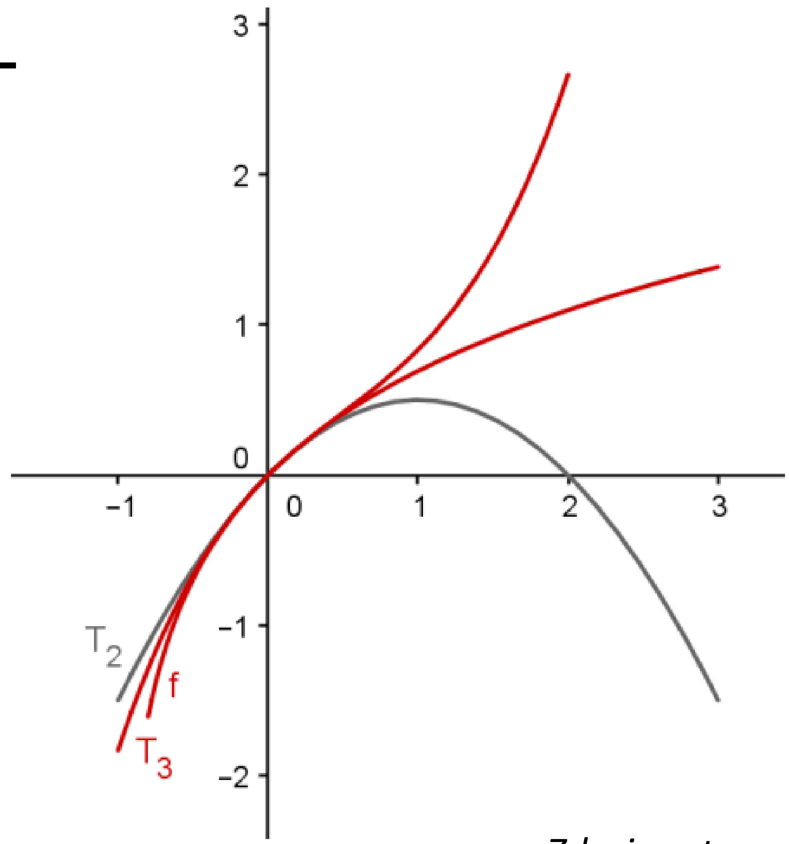
$$c = 0, f(x) = \ln(1 + x)$$

$$T_n(x) = \sum_{k=1}^{n} (-1)^{k-1} \frac{x^k}{k}$$

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$$c = 0, f(x) = \ln(1 - x)$$

$$T_n(x) = \sum_{k=1}^{n} (-1)^{k-1} \frac{(-x)^k}{k} = -\sum_{k=1}^{n} \frac{x^k}{k}$$

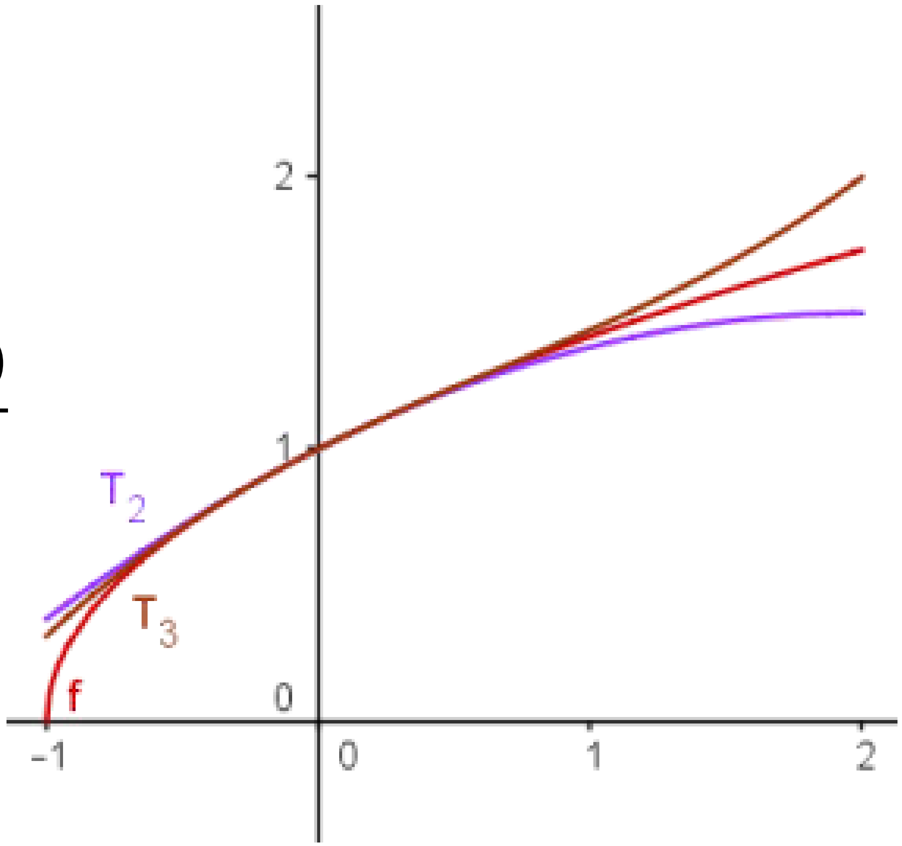
$$\ln\frac{1+x}{1-x} = \ln(1 + x) - \ln(1 - x) \,,\, x \in (-1,1)$$

$$T_n(x) = 2\sum_{k=1}^{n} \frac{x^{2k-1}}{2k - 1}$$

$$C = 0,\ f(x) = (1 + x)^m$$

$$T_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^k$$

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}$$

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$$\int_0^x e^{-x^2} dx \approx \int_0^x \left(1 - x^2 + \frac{x^4}{2} - \frac{x^6}{3!} + \frac{x^8}{4!} - \frac{x^{10}}{5!} + \frac{x^{12}}{6!}\right) dx$$

$$\approx \left(x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \frac{x^9}{4!9} - \frac{x^{11}}{5!11}\right) + \frac{x^{13}}{6!13}$$

Pro $x = 1$ zaokrouhlit na 3 desetinná místa:

$$\frac{1}{42} = 0{,}023\,80\ldots, \frac{1}{4!\,9} = 0{,}004\,62\ldots, \frac{1}{5!\,11} = 0{,}000\,75\ldots, \frac{1}{6!\,13} = 0{,}000\,04$$

Zaokrouhlovací chyby se sčítají, je třeba zaokrouhlovat na 4 místa:

$$\int_0^1 e^{-x^2} dx \approx 1 - 0{,}333\,3 + 0{,}1 - 0{,}023\,8 + 0{,}004\,6 - 0{,}000\,8 \approx 0{,}747$$

Řešení diferenciální rovnice: $(1-x^2)y'' - 4xy' - 2y = 0$

$$y = \sum_{n=0}^{+\infty} a_n x^n, y' = \sum_{n=1}^{+\infty} n a_n x^{n-1}, y'' = \sum_{n=2}^{+\infty} n(n-1) a_n x^{n-2}$$

$$\sum_{n=2}^{+\infty} n(n-1) a_n x^{n-2} - \sum_{n=2}^{+\infty} n(n-1) a_n x^n - 4\sum_{n=1}^{+\infty} n a_n x^n - 2\sum_{n=0}^{+\infty} a_n x^n = 0$$

Koeficient u $x^n$: $(n+2)(n+1)a_{n+2} - n(n-1)a_n - 4na_n - 2a_n = 0$

$$a_{n+2} = a_n$$
$$a_{2k} = a_0, a_{2k+1} = a_1$$

$$y = a_0 + a_1 x + a_0 x^2 + a_1 x^3 + \cdots = a_0 \sum_{k=0}^{+\infty} x^{2k} + a_1 \sum_{k=0}^{+\infty} x^{2k+1}$$

Řešení rovnice: $y = \frac{a_0}{1-x^2} + \frac{a_1 x}{1-x^2}$ , $|x| < 1$